1 a) Assume that $x \neq 0$. Then we may rewrite the equation as

$$y' + (x^{-2} - x^{-1}) y = 0. \quad (*)$$

This is a linear 1st order ODE, and the integrating factor is

$$\mu(x) = e^{\int (x^{-2} - x^{-1}) dx} = e^{-x^{-1} - \ln|x|}$$

Since we are interested in the solution that is defined at $x = 1$, we will assume $x > 0$. Thus

$$\mu(x) = e^{-\frac{1}{x}} (\underbrace{e^{\ln x}}_{= x})^{-1} = \frac{1}{x} e^{-\frac{1}{x}}$$

Multiplying $(*)$ by $\mu(x)$ we get

$$\frac{1}{x} e^{-\frac{1}{x}} y' + \underbrace{(x^{-3} - x^{-2}) e^{-\frac{1}{x}}}_{(\star)} y = 0.$$

This brings the equation in the form

$$\underline{\left(\frac{1}{x} e^{-\frac{1}{x}} y\right)' = 0} \quad \text{as}$$

$$\left(\frac{1}{x} e^{-\frac{1}{x}}\right)' = -\frac{1}{x^2} e^{-\frac{1}{x}} + \frac{1}{x^3} e^{-\frac{1}{x}}$$

which is precisely $(\star)$.

Integrating (⊙) we find

$\frac{1}{x}e^{-\frac{1}{x}}y = C$ ← const $\Rightarrow \boxed{y = Cxe^{\frac{1}{x}}}$

$y(1) = -1$ gives $-1 = Ce \Rightarrow C = -\frac{1}{e}$

so $y = -\frac{1}{e}xe^{\frac{1}{x}}$.

Control: $y(1) = -\frac{1}{\cancel{e}} \cdot 1\cancel{e}^{1} = -1$ $\boxed{\text{OK!}}$

$x^2y' + y = xy \Leftrightarrow$ / multiply by $-e$ /

$x^2(xe^{\frac{1}{x}})' + xe^{\frac{1}{x}} = x^2e^{\frac{1}{x}} \Leftrightarrow$

$x^2(\cancel{e^{\frac{1}{x}}} - \cancel{\frac{1}{x}e^{\frac{1}{x}}}) + \cancel{xe^{\frac{1}{x}}} = \cancel{x^2e^{\frac{1}{x}}}$ $\boxed{\text{OK!}}$

b) The solution is defined for $x > 0$.

$\boxtimes$

② a) Let $M(x,y) = 6xy - y^3$
$N(x,y) = 4y + 3x^2 - 3xy^2.$

The equation is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \iff 6x - 3y^2 = 6x - 3y^2$$

true!

b) To solve the equation we need to find a function $f = f(x,y)$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = M(x,y) \\ \dfrac{\partial f}{\partial y} = N(x,y) \end{cases} \iff \begin{cases} \dfrac{\partial f}{\partial x} = 6xy - y^3 & (1) \\ \dfrac{\partial f}{\partial y} = 4y + 3x^2 - 3xy^2 & (2) \end{cases}$$

From (1) we find that

$$f(x,y) = \int (6xy - y^3)\,dx + g(y)$$
$$= 3x^2y - y^3x + g(y)$$

Inserting this in (2) we see that

$$\cancel{3x^2} \cancel{- 3y^2x} + g'(y) = 4y \cancel{+3x^2} \cancel{-3xy^2}$$

Hence $g'(y) = 4y$, so we can take

$g(y) = 2y^2 \Rightarrow \underline{f(x,y) = 3x^2y - y^3x + 2y^2}.$

As a consequence, our equation is of the form

$$df = 0 \iff \underset{\uparrow\text{const}}{f(x,y) = C} \quad \text{or}$$

$$\boxed{3x^2y - y^3x + 2y^2 = C} \quad (*)$$

$(*)$ defines $y = y(x)$ implicitly. $\boxtimes$

(3) a) This is a linear 2nd order ODE with constant coefficients.
The characteristic equation is

$$r^2 + 9 = 0 \Leftrightarrow r_{1,2} = \pm 3i.$$

The general solution is

$$\boxed{y = C_1 \cos 3x + C_2 \sin 3x} \qquad (H)$$

b) Looking at the RHS we see that we should seek a particular solution in the form

$$y_p = x^s (A \cos 3x + B \sin 3x)$$

where $s \in \{0, 1, 2\}$.

Clearly $s=0$ will not work, because y_p has in this case the same form as in (H) and will therefore be the solution of the homogeneous problem.

We try $s=1$. In this case

$$y_p = x(A \cos 3x + B \sin 3x)$$

$$y_p' = A \cos 3x + B \sin 3x + x(-3A \sin 3x + 3B \cos 3x)$$

$$y_p'' = -3A\sin 3x + 3B\cos 3x + (-3A\sin 3x + 3B\cos 3x) + x(-9A\cos 3x - 9B\sin 3x)$$

Thus $y_p'' + 9y_p = -6A\sin 3x + 6B\cos 3x$

This should be equal to $2\cos 3x + 3\sin 3x$ so we set

$$\begin{matrix} -6A = 3 \\ 6B = 2 \end{matrix} \iff \begin{cases} A = -\frac{1}{2} \\ B = \frac{1}{3} \end{cases}$$

All in all, $y_p = x\left(-\frac{1}{2}\cos 3x + \frac{1}{3}\sin 3x\right)$

and $y = y_h + y_p$, where y_h is the solution of the homogeneous problem obtained in a). Thus

$$\boxed{y = C_1\cos 3x + C_2\sin 3x + x\left(-\frac{1}{2}\cos 3x + \frac{1}{3}\sin 3x\right).}$$

(4) a) We compute $p(x) = \frac{2}{4x} = \frac{1}{2x}$
and $q(x) = -\frac{1}{4x}$

Both functions $\to \pm\infty$ as $x \to 0$ so
$x=0$ is a singular point.

As $xp(x) = \frac{1}{2}$ and $x^2 q(x) = -\frac{1}{4}x$
are polynomials and hence analytic
functions we conclude that $x=0$
is a regular singular point.

b) $xp(x) = \frac{1}{2}$ gives $p_0 = \frac{1}{2}$
$x^2 q(x) = -\frac{1}{4}x$ gives $q_0 = 0$ } as coeff. of the indicial equation

The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = 0 \iff$$

$$r^2 - \frac{1}{2}r = 0 \iff r(r - \tfrac{1}{2}) = 0$$

so $r_1 = 0$, $r_2 = \frac{1}{2}$.

c) We seek a solution corresponding
to $r_2 = \frac{1}{2}$ in the form

$$y = x^{1/2} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}}.$$

In this case $y' = \sum_{n=0}^{\infty} c_n (n+\frac{1}{2}) x^{n-1/2}$

$$y'' = \sum_{n=0}^{\infty} c_n (n+\tfrac{1}{2})(n-\tfrac{1}{2}) x^{n-\frac{3}{2}}, \text{ so}$$

the equation $4xy'' + 2y' - y = 0$ becomes

$$\sum_{n=0}^{\infty} 4c_n (n+\tfrac{1}{2})(n-\tfrac{1}{2}) x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} 2c_n (n+\tfrac{1}{2}) x^{n-1/2}$$
$$- \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} = 0 \quad \Leftrightarrow$$

$$\sum_{n=0}^{\infty} c_n \underbrace{(4n^2 - \cancel{1} + 2n + \cancel{1})}_{=0 \text{ when } n=0} x^{n-1/2} - \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} = 0$$

$$\sum_{n=1}^{\infty} c_n (4n^2 + 2n) x^{n-1/2} - \sum_{n=0}^{\infty} c_n x^{n+1/2} = 0$$

Shifting the index in the first sum we get

$$\sum_{n=0}^{\infty} c_{n+1} (\underbrace{4(n+1)^2 + 2(n+1)}_{=2(n+1)(2n+2+1)}) x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} = 0$$

or $$\sum_{n=0}^{\infty} \left[(2n+2)(2n+3) c_{n+1} - c_n \right] x^{n+\frac{1}{2}} = 0$$

By the identity principle we have

$$\boxed{c_{n+1} = \frac{c_n}{(2n+2)(2n+3)}}.$$

We let $c_0 = 1$, then

$$c_1 = \frac{c_0}{2 \cdot 3} = \frac{1}{2 \cdot 3} = \frac{1}{3!}$$

$$c_2 = \frac{c_1}{4 \cdot 5} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{5!}$$

$$c_3 = \frac{c_2}{6\cdot 7} = \frac{1}{5!\,6\cdot 7} = \frac{1}{7!} \dots$$

Clearly $\boxed{c_n = \frac{1}{(2n+1)!}}$

Our series is thereby

$$y = \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} \iff$$

$$y = \sum_{n=0}^{\infty} \frac{x^{\frac{2n+1}{2}}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(\sqrt{x})^{2n+1}}{(2n+1)!}$$

Comment: by looking at known Maclaurin series, we we see that $y = \sinh\sqrt{x}$, and it is very easy to check that it is indeed the solution of the equation:

$$(\sinh\sqrt{x})' = \cosh\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$(\sinh\sqrt{x})'' = \sinh\sqrt{x} \cdot \frac{1}{4x} - \cosh\sqrt{x} \cdot \frac{1}{4x\sqrt{x}}$$

so $4x(\sinh\sqrt{x})'' + 2(\sinh\sqrt{x})' - \sinh\sqrt{x} =$

$$= \cancel{\sinh\sqrt{x}} - \cancel{\frac{\cosh\sqrt{x}}{\sqrt{x}}} + \cancel{\frac{\cosh\sqrt{x}}{\sqrt{x}}} - \cancel{\sinh\sqrt{x}}$$

$= 0$ $\boxtimes$

(5) a) The system's matrix is

$$A = \begin{pmatrix} 6 & 1 \\ -1 & 4 \end{pmatrix}.$$

The eigenvalues are found by solving

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 6-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\iff (6-\lambda)(4-\lambda) + 1 = 0$$

$$\iff \lambda^2 - 10\lambda + \underbrace{24 + 1}_{=25} = 0$$

$$\iff (\lambda - 5)^2 = 0$$ which gives $\underline{\underline{\lambda_1 = \lambda_2 = 5}}$

The respective eigenvector $K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ is found by solving

$$(A - \lambda I)K = 0 \iff \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} k_1 + k_2 = 0 \\ \cancel{-k_1 - k_2 = 0} \end{cases} \iff k_2 = -k_1$$

We can take $K = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

One solution of the system is

$$\underline{X_1 = e^{\lambda t} K = e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$$

To find another solution, we solve

$(A-\lambda I)P = K$ for $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, i.e.

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Leftrightarrow p_1 + p_2 = 1$$

so we can take $p_1 = 1$, $p_2 = 0$ and $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Another solution is

$$X_2 = e^{\lambda t}(Kt + P) = e^{5t}\begin{pmatrix} t+1 \\ -t \end{pmatrix}$$

The general solution is

$$X = C_1 X_1 + C_2 X_2 =$$
$$= C_1 e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} t+1 \\ -t \end{pmatrix}$$

b) To sketch the phase portrait, it is convenient to rewrite the solution in the form

$$X = C_1 e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 t e^{5t}\begin{pmatrix} 1 + \frac{1}{t} \\ -1 \end{pmatrix}$$

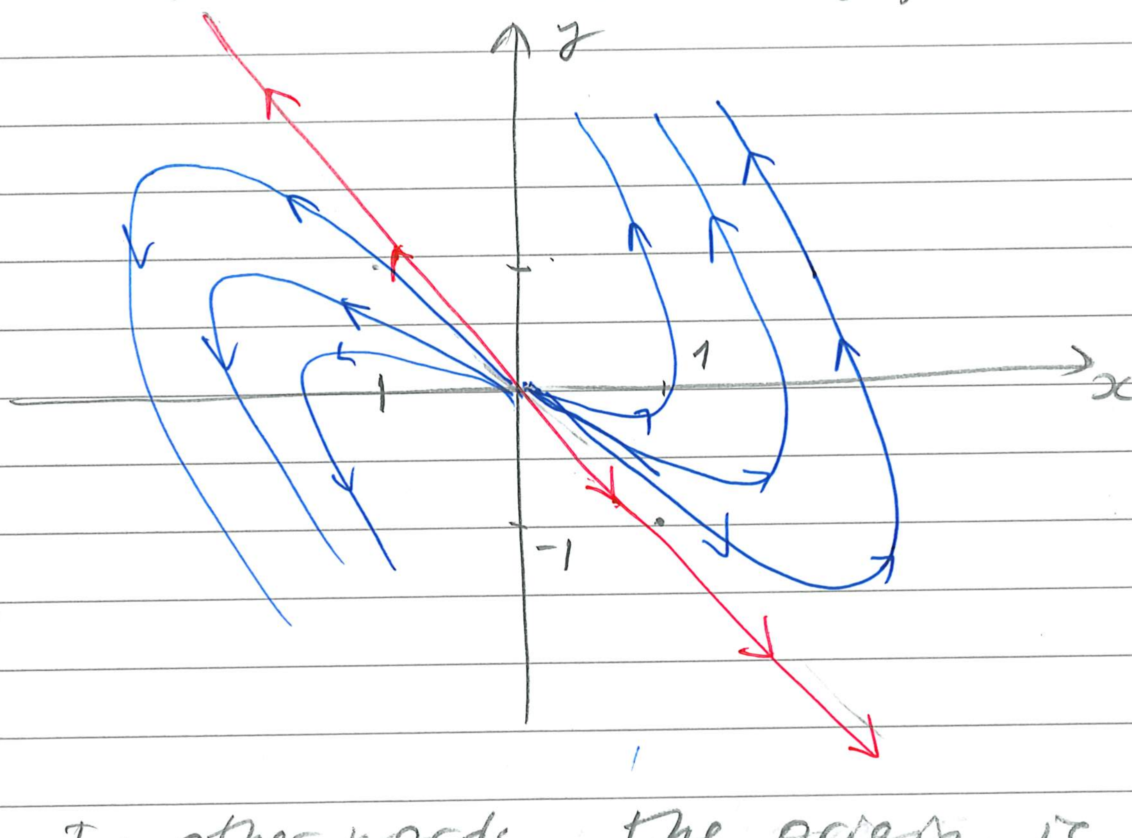
When $C_2 = 0$ we get the straight line parallel to the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ through the origin.

When $c_2 \neq 0$, the second term will dominate as $t \to \infty$, and we can see that

$$X \approx c_2 t e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i.e. trajectories become parallel to the line. When $t \to -\infty$, a simple limit computation shows that $X \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This gives us the following picture:



In other words, the origin is an asymptotically unstable improper node.

(6) We use the method of variation of parameters. For this we first solve the homogeneous system

$$X' = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{=A} X. \qquad (H)$$

looking for eigenvalues

$$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda_{1,2} = \pm i$ and

looking for eigenvectors

$$(A - iI)K = 0 \Leftrightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

so we may take $K = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Consequently, the complex solution of (H) is

$$X = e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (\cos t + i \sin t)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

The general solution of (H) is

$$X_h = C_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

The fundamental matrix of the system is

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

and we may find a particular solution of the non-homogeneous system by using the formula

$$X_p = \Phi(t) \int \Phi^{-1}(t) F(t)\, dt$$

where $F(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

For this, we need to compute

$$\Phi^{-1}(t) = \frac{1}{\underbrace{-\cos^2 t - \sin^2 t}_{=-1}} \begin{pmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

Then

$$\Phi^{-1}(t) F(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 t + \sin^2 t \\ \sin t \cos t - \cos t \sin t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with $\int \Phi^{-1}(t) F(t)\, dt = \begin{pmatrix} t \\ 0 \end{pmatrix}$

Finally,

$$X_p = \Phi(t)\int \Phi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} t\cos t \\ t\sin t \end{pmatrix}$$

The general solution of the system is

$$X = X_h + X_p = \begin{pmatrix} C_1\cos t + C_2\sin t + t\cos t \\ C_1\sin t - C_2\cos t + t\sin t \end{pmatrix}$$

$\boxtimes$

7 a) We set $x = u$, $y = u'$ and obtain

$$y' = u'' = -u' - 2u - u^2 = -y - 2x - x^2 .$$

The respective system is

$$\boxed{\begin{cases} x' = y \\ y' = -y - 2x - x^2 \end{cases}}$$

b) The critical points are found by solving

$$\begin{cases} y = 0 \\ -y - 2x - x^2 = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x(x+2) = 0 \end{cases}$$

which gives $(0,0)$ and $(-2,0)$.

c) For the classification of critical points we need to compute the Jacobian of the system

$$J(x,y) = \begin{pmatrix} \partial_x(y) & \partial_y(y) \\ \partial_x(-y-2x-x^2) & \partial_y(-y-2x-x^2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2-2x & -1 \end{pmatrix}.$$

At $(0,0)$ we have $J(0,0) = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$

The eigenvalues are found by solving

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + \lambda + 2 = 0$$

$$\Leftrightarrow \left(\lambda + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

so $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Consequently, $(0,0)$ is an asymptotically stable spiral point.

At $(-2, 0)$ we have

$$J(-2,0) = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$$ so

$$\begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + \lambda - 2 = 0$$

$$\Leftrightarrow \lambda_1 = 1, \; \lambda_2 = -2$$

Consequently, $(-2, 0)$ is an unstable saddle point.

8) a) $W(x,y) = -(x+2y)^2$
satisfies $W(0,0) = 0$ and
$W(x,y) \leq 0$ for all x and y
so it is negative semidefinite.

However it is not negative definite
as $W(x,y) = 0 \iff x = -2y$ so
we do not have $W(x,y) < 0$ for
all $(x,y) \neq (0,0)$ that are sufficiently
close to the origin.

As for the other two functions,
we see that $U(x,y)$ is positive
definite and $V(x,y)$ is indefinite,
i.e. it takes both positive and
negative values in any neighborhood
of $(0,0)$.

b) All we need is to find a function
$V(x,y) = ax^k + by^l$ which is positive
at least at one point in any
neighborhood of $(0,0)$ and such
that

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}(x^9 - y^5) + \frac{\partial V}{\partial y}(-x^5 - y^7)$$

is positive definite.

We compute:

$$\frac{\partial V}{\partial t} = akx^{k-1}(x^9 - y^5) + bly^{l-1}(-x^5 - y^7)$$

$$= akx^{k+8} - akx^{k-1}y^5 - bly^{l-1}x^5 - bly^{l+6}$$

Mixed terms (with both x and y) are usually those that change sign, so we will get rid of them by requesting that

$$akx^{k-1}y^5 = -bly^{l-1}x^5.$$

This gives $\begin{cases} k-1=5 \\ l-1=5 \\ b=-a \end{cases} \iff \begin{cases} k=l=6 \\ b=-a \end{cases}$.

Let us set $a=1$, then $b=-1$ and

$$\frac{\partial V}{\partial t} = 6x^{14} \cancel{- 6x^5y^5 + 6y^5x^5} + 6y^{12}$$
$$= 6(x^{14} + y^{12}).$$

Clearly, this is positive definite

as $\frac{\partial V}{\partial t} \geq 0$ for all x and y and

$\frac{\partial V}{\partial t} = 0$ if and only if $x = y = 0$.

Further, $V(x,y) = x^6 - y^6$ satisfies $V(x,0) = x^6 > 0$ for any $x \neq 0$ so it takes positive values in any nbhd of $(0,0)$. This confirms that $(0,0)$ is unstable