UPPSALA UNIVERSITET
Matematiska institutionen
Anna Sakovich

Prov i matematik
KandMa, KandFy, STS
1MA032 ODE I
2020-08-24

Writing time: 08:00 – 13:00. Submission time: not later than 13:20. This is an Open Book Exam which means that you may use the textbook, your notes and/or other resources. The credit for each problem is shown below. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be accompanied with explanatory text. Please write your name and page number on each page. The examiner can be reached at 0760480930 *or by email* anna.sakovich@math.uu.se *during the writing time.*

1. Consider the equation

$$\frac{dy}{dx} = y^2 - 4.$$

(a) (2 points) Determine all constant solutions of the equation. Without solving the equation, sketch a few other solution curves.

(b) (3 point) Solve the equation.

2. (a) (1 points) Find the value of the constant parameter k so that the differential equation

$$(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3 + 6y^2)dy = 0$$

is exact.

(b) (4 points) For this value of k, find the solution satisfying the initial condition $y(1) = -1$. Note that the solution may be defined implicitly, i.e. in the form $F(x, y) = 0$.

3. (a) (1 points) Verify that $y_1 = e^x$ is a solution of the equation

$$(x + 1)y'' - (x + 2)y' + y = 0, \qquad x > -1.$$

(b) (4 points) Find the general solution of the equation in (a).

4. (a) (1 point) Show that $x = 0$ is an ordinary point of the differential equation

$$(x^2 + 1)y'' + xy' - y = 0.$$

(b) (1 point) Without solving the equation, find a lower bound for the radius of convergence of power series solutions about $x = 0$.

(c) (3 points) Find the power series solution about the point $x = 0$.

Please turn the page!

5. (a) (3 points) Find the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{X}.$$

(b) (2 points) Sketch the phase portrait of the system.

6. (5 points) Find the general solution of the problem

$$\mathbf{X}'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X}(t) + \begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}.$$

7. A pendulum-like physical system is described by the equation

$$u'' + \sin u = 0,$$

where $u = u(t)$, $t \in \mathbb{R}$.

(a) (1 point) Rewrite the equation as a first order system.

(b) (1 point) Find all the critical points of the system.

(c) (3 points) Use the Jacobian matrix to classify (if possible) the critical points of the system according to their type and stability.

8. (5 points) Show that $(0, 0)$ is a stable critical point of the system

$$\begin{aligned} x' &= y - x^3y^4 - x^5y^2 \\ y' &= -x - x^4y^3 - x^2y^5 \end{aligned}$$

Hint: look for the Liapunov function in the form $V(x, y) = ax^2 + by^2$.

GOOD LUCK!