UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
1MA032
ODE I
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Allowed aids: writing materials and the book of Boyce-DiPrime entitled 'Elementary differential equations and boundary value problem' used for the course. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5, respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions must be accompanied with explanatory text.

1. Determine the solution of the differential equation

$$(y^2e^x - 1)dx + 2ye^x dy = 0$$

which satisfies the initial condition $y(0) = 1$.

2. Find the general solution $y = y(x)$ to the ODE

$$(a) \quad 3x^2y'' + 6xy' - 6y = 0,$$

and the general solution $y = y(x)$ to the ODE

$$(b) \quad y'' + 4y' + 4y = e^{-2x} \ln(x).$$

3. Consider the system of ODEs

$$x'(t) = x + 2y - 2e^t$$
$$y'(t) = 4x + 3y - 10e^t$$

where $x = x(t)$ and $y = y(t)$, on the interval $-\infty < t < \infty$.

- Find the general solution to the system of ODEs.
- Classify which type of critical point the origin $(0, 0)$ is for the homogeneous part of the system, with respect to both portrait type and stability.
- Sketch a phase portrait for the system, including at least 4 different representative trajectories.

4. Find a nontrivial solution of the differential equation

$$x^2y'' + x(x^2 - 3)y' + 4y = 0.$$

Express the solution in terms of elementary functions.

5. Consider the system

$$x'(t) = x(3 - x - 2y)$$
$$y'(t) = y(5 - 4x - y)$$

where $x = x(t)$ and $y = y(t)$, on the interval $-\infty < t < \infty$.

- Determine all the critical points of the system.
- Is the origin an *isolated* critical point of the system? Justify.
- Classify the origin $(0,0)$ with regards to local portrait type and stability (as $t \to \infty$) for this system.

6. Prove that $(0,0)$ is the only critical point for the system

$$
\begin{aligned}
x'(t) &= -y^3 - x^3 \\
y'(t) &= x^5 y^2 - y.
\end{aligned}
$$

Use the Lyapunov method to determine the stability properties of $(0,0)$.

7. Determine a linear ODE having $\{e^{2x}, e^{x^2/2}\}$ as its fundamental set.

8. Prove that the initial value problem

$$x' = x^2 + \cos(t^2),\ x(0) = 0,$$

has a unique solution on $[-1/2, 1/2]$ and that $|x(t)| \leq 1$ whenever $t \in [-1/2, 1/2]$. (Hint: recall the Picard iteration).

GOOD LUCK!

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