Exam ODE I; 1MA032, 2021-01-08.

1. $y^2 = e^{-x}(1+x)$

2. a) $y(x) = c_1 x + c_2 x^{-2}$, $c_1, c_2 \in \mathbb{R}$.

b) $$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 e^{-2x}\left(\frac{\ln x}{2} - \frac{3}{4}\right)$$

3. General solution

$$Y(t) = c_1 \begin{pmatrix}1\\2\end{pmatrix} e^{5t} + c_2 \begin{pmatrix}1\\-1\end{pmatrix} e^{-t} + \begin{pmatrix}2\\1\end{pmatrix} e^{t}$$

(0,0) is a saddle point, unstable.

4. $$y(x) = x^2 e^{-x^2/2} = x^2 \sum_{k=0}^{\infty} \frac{(-x^2/2)^k}{k!} = x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-k} x^{2k}}{k!}$$

is a solution.

5. The critical points are $(0,0), (0,5), (3,0), (1,1)$. $(0,0)$ is isolated as $B_\varepsilon(0,0)$ for $\varepsilon > 0$ small does not contain any critical point beyond $(0,0)$. By using the theory of locally linear systems one concludes that $(0,0)$ is an unstable node for the non-linear system. The eigenvalues of $J(0,0)$ are 3 and 5, can be used to picture local dynamics in neighbourhoods of $(0,0)$.

6. Critical points

$$\begin{cases} -y^3 - x^3 = 0 & \Rightarrow \quad y = -x \\ x^5 y^2 - y = 0 & \Rightarrow \quad x^7 + x = x(x^6+1) = 0 \end{cases}$$

$\Rightarrow x = 0 = y$.

Let $V(x,y) = ax^6 + by^2 = x^6 + 3y^2$.

Then $\dot{V} = -6x^4 - by^2$.

$\therefore V$ pos. def, $\dot{V}$ neg. def.

Lyapunov's stability theorem

$\Rightarrow (0,0)$ asymptotically stable c.p.

7. An example of such an ODE is

$$(x-2)y'' - (x^2-3)y' + 2(x-1)^2 y = 0.$$

which can be found by the ansatz

$y'' + p(x)y' + q(x)y = 0$ and by plugging $e^{2x}, e^{x^2/2}$ into the eq.

8.

$$(*)\quad x'(t) = F(x(t), t) \quad x(0) = 0 \text{ where}$$

$F(x,t) = x^2 + \cos t^2$.
F is a smooth function of (x,t) and it partial derivatives are bounded on $[-1/2, 1/2]$. By Picard iteration one realizes, or by the general existence theorem, that $(*)$ has a unique solution on $[-1/2, 1/2]$
The Picard iteration reads

$$x_0(t) \equiv x(0) = 0,$$

$$x_{k+1}(t) = x_k(0) + \int_0^t F(x_k(\tau), \tau)\, d\tau.$$

$$= \int_0^t F(x_k(\tau), \tau)\, d\tau.$$

we know that

$$X_n(t) \to X(t) \qquad \forall t \in [-1/2, 1/2]$$

as $n \to \infty$ and where $X(t)$ is the unique solution to $(*)$.

We now prove that $|X_n(t)| \leq 1$ whenever $t \in [-1/2, 1/2]$. As a conseq. $|X(t)| \leq 1 \quad \forall t \in [-1/2, 1/2]$.

Obviously $|X_0(t)| \equiv 0 \leq 1$.

Assume

$$|X_k(t)| \leq 1 \qquad \forall t \in [-1/2, 1/2].$$

Then

$$|X_{k+1}(t)| \leq \int_0^t |F(X_k(\tau), \tau)| dt$$

$$\leq \int_0^t (|X_k(\tau)|^2 + \cos \tau^2) d\tau$$

$$\leq \int_0^t (1+1) d\tau = 2t \leq 1$$

as $|t| \leq 1/2$: Here I considered $t \geq 0$. $t \leq 0$ is handled analogously.