UPPSALA UNIVERSITET
Matematiska institutionen
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1MA032 Ordinary differential equations I
MaKand, FyKand, STS, fristående
2021–06–05

Allowed aids: writing materials. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5, respectively, one should obtain at least 18, 25 and 32 point, respectively. Solutions must be accompanied with explanatory text

1. Solve the initial value problem

$$\frac{1}{\cos(x)}y' - \frac{e^{\sin^2(x)}}{\cos(x)} = 2\sin(x)y \quad y(0) = 1.$$

(5 points)

2. Find the general solution $y = y(x)$ of the equation

$$y'' - 4y' + 4y = e^{2x} + 3\cos x + 2\sin x.$$

(5 points)

3. Find the general solution $y = y(x)$ of the equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{1 + x^2}.$$

(5 points)

4. Consider the differential equation

$$x^2y'' + (x^2 + x)y' - 2y = 0.$$

a) Show that this equation has a regular singular point at $x = 0$.

b) Determine the indicial equation and its roots.

c) Find a series solution for $x > 0$ corresponding to the larger root of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients.

(5 points)

5. Find the general solution to the system

$$\begin{cases} x' &= 3x + 5y \\ y' &= x - y \end{cases}$$

and sketch the phase portrait.

(5 points)

Please turn the page!

6. Consider the system

$$\begin{cases} x' &= -x - 2y + \dfrac{e^t}{1-e^{2t}} \\ y' &= y \end{cases}.$$

a) Given an initial value $(x(t_0), y(t_0)) = (x_0, y_0)$, what does the existence and uniqueness theorem for linear systems say about on which interval the solution is defined?

b) Find the general solution to the system.

(5 points)

7. The van der Pol equation is given by

$$u'' - \mu(1-u^2)u' + u = 0,$$

for a parameter $\mu \in \mathbb{R}$.

a) Rewrite the equation as a system of first order equations.

b) Show that the origin is the only critical point of the system.

c) Determine the type and stability of the origin for $\mu \neq 0$. For $\mu = \pm 2$ it's enough to only give the stability and you don't have to determine the type.

(5 points)

8. Consider the system

$$\begin{cases} x' &= 2x^3 + 2y^3 \\ y' &= x^3 - e^x y^7 \end{cases}.$$

a) Show that the origin is a critical point and that it's *isolated*.

b) Determine the stability of the origin. *Hint*: Search for a suitable Lyapunov function of the form $V(x,y) = ax^k + bx^l$.

(5 points)