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1MA032 Ordinary differential equations I
MaKand, FyKand, STS, fristående
2021–06–05

Allowed aids: writing materials. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5, respectively, one should obtain at least 18, 25 and 32 point, respectively. Solutions must be accompanied with explanatory text

1. Solve the initial value problem

$$\frac{1}{\cos(x)}y' - \frac{e^{\sin^2(x)}}{\cos(x)} = 2\sin(x)y \quad y(0) = 1.$$

(5 points)

2. Find the general solution $y = y(x)$ of the equation

$$y'' - 4y' + 4y = e^{2x} + 3\cos x + 2\sin x.$$

(5 points)

3. Find the general solution $y = y(x)$ of the equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{1 + x^2}.$$

(5 points)

4. Consider the differential equation

$$x^2y'' + (x^2 + x)y' - 2y = 0.$$

a) Show that this equation has a regular singular point at $x = 0$.

b) Determine the indicial equation and its roots.

c) Find a series solution for $x > 0$ corresponding to the larger root of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients.

(5 points)

5. Find the general solution to the system

$$\begin{cases} x' &= 3x + 5y \\ y' &= x - y \end{cases}$$

and sketch the phase portrait.

(5 points)

Please turn the page!

6. Consider the system

$$\begin{cases} x' &= -x - 2y + \dfrac{e^t}{1-e^{2t}} \\ y' &= y \end{cases}.$$

a) Given an initial value $(x(t_0), y(t_0)) = (x_0, y_0)$, what does the existence and uniqueness theorem for linear systems say about on which interval the solution is defined?

b) Find the general solution to the system.

(5 points)

7. The van der Pol equation is given by

$$u'' - \mu(1-u^2)u' + u = 0,$$

for a parameter $\mu \in \mathbb{R}$.

a) Rewrite the equation as a system of first order equations.

b) Show that the origin is the only critical point of the system.

c) Determine the type and stability of the origin for $\mu \neq 0$. For $\mu = \pm 2$ it's enough to only give the stability and you don't have to determine the type.

(5 points)

8. Consider the system

$$\begin{cases} x' &= 2x^3 - 2y^3 \\ y' &= x^3 + e^x y^7 \end{cases}.$$

a) Show that the origin is a critical point and that it's *isolated.*

b) Determine the stability of the origin. *Hint*: Search for a suitable Lyapunov function of the form $V(x, y) = ax^k + by^l$.

(5 points)

Solutions to exam in 1MA032 Ordinary differential equations I 2021–06–05

Solution to problem 1. We start by noticing that the equation is of first order and linear in y, so we can solve it using an integrating factor. We rewrite it as

$$\begin{aligned}
&\frac{1}{\cos(x)}y' - \frac{e^{\sin^2(x)}}{\cos(x)} = 2\sin(x)y \\
\iff &\frac{1}{\cos(x)}y' - 2\sin(x)y = \frac{e^{\sin^2(x)}}{\cos(x)} \\
\iff &y' - 2\sin(x)\cos(x)y = e^{\sin^2(x)}.
\end{aligned}$$

The integrating factor is

$$\mu(x) = e^{\int -2\sin x\cos x \; dx},$$

and the integral is given by $\cos^2 x$, which can easily be checked and computed using for example the change of variables $u = \cos x$. This gives us the integrating factor $\mu(x) = e^{\cos^2 x}$. Multiplying the equation with that we get

$$e^{\cos^2 x}y' - 2\sin(x)\cos(x)e^{\cos^2 x}y = e^{\sin^2(x)}e^{\cos^2 x}.$$

The left hand side is given by $\left(e^{\cos^2 x}y\right)'$ and the right hand side by $e^{\sin^2 + \cos^2 x} = e^1 = e$. This reduces the equation to

$$\left(e^{\cos^2 x}y\right)' = e.$$

Integrating both sides we arrive at

$$e^{\cos^2 x}y = ex + C \iff y = e^{-\cos^2 x}(ex + C)$$

for some constant C.

The initial value $y(0) = 1$ gives us

$$1 = e^{-\cos^2 0}(e \cdot 0 + C) = e^{-1}C,$$

so $C = e$. The solution to the initial value problem is hence given by

$$y = e^{-\cos^2 x}(ex + e) = e^{1-\cos^2 x}(x + 1) = e^{\sin^2 x}(x + 1).$$

Solution to problem 2. We start by finding the general solution to the associated homogeneous system $y'' - 4y' + 4y = 0$. The characteristic equation is given by $r^2 - 4r + 4 = 0$ which has the double root $r_{1,2} = 2$. The general solution to the homogeneous equation is thus given by

$$y_h = C_1e^{2x} + C_2xe^{2x} = e^{2x}(C_1 + C_2x).$$

To find a particular solution we split the right hand side into two parts, e^{2x} and $3\cos x + 2\sin x$. For e^{2x} a natural choice for the particular solution would be De^{2x}, however that solves the homogeneous equation and so does Dxe^{2x}, we therefore make the guess $y_{p,1} = Dx^2e^{2x}$. This gives us

$$\begin{aligned}
y_{p,1}' &= D2xe^{2x} + D2x^2e^{2x}, \\
y_{p,1}'' &= D2e^{2x} + D8xe^{2x} + D4x^2e^{2x}
\end{aligned}$$

which after insertion into the equation yields

$$D2e^{2x} + D8xe^{2x} + D4x^2e^{2x} - 4(D2xe^{2x} + D2x^2e^{2x}) + 4Dx^2e^{2x} = e^{2x}.$$

Most terms on the left hand side cancel out and we are left with $D2e^{2x} = e^{2x}$, hence $D = \dfrac{1}{2}$.

For $3\cos x + 2\sin x$ a natural guess is $y_{p,2} = A\cos x + B\sin x$, giving us

$$\begin{aligned} y'_{p,2} &= -A\sin x + B\cos x, \\ y''_{p,2} &= -A\cos x - B\sin x \end{aligned}$$

and insertion gives

$$-A\cos x - B\sin x - 4(-A\sin x + B\cos x) + 4(A\cos x + B\sin x) = 3\cos x + 2\sin x.$$

Collecting the terms on the left hand side we arrive at

$$(3A - 4B)\cos x + (4A + 3B)\sin x = 3\cos x + 2\sin x.$$

We thus get the linear system

$$\begin{cases} 3A - 4B &= 3 \\ 4A + 3B &= 2 \end{cases},$$

which can be checked to have the solutions $A = \dfrac{17}{25}$, $B = -\dfrac{6}{25}$.

Putting all of this together we get that the general solution is given by

$$y = y_h + y_{p,1} + y_{p,2} = e^{2x}(C_1 + C_2x) + \frac{1}{2}x^2e^{2x} + \frac{17}{25}\cos x - \frac{6}{25}\sin x.$$

Solution to problem 3. We start by finding the general solution to the associated homogeneous system $y'' - 6y' + 9y = 0$. The characteristic equation is given by $r^2 - 6r + 9 = 0$ which has the double root $r_{1,2} = 3$. The general solution to the homogeneous equation is thus given by

$$y_h = C_1e^{3x} + C_2xe^{3x} = e^{3x}(C_1 + C_2x).$$

To find a particular solution we notice that the right hand side is not on simple form for which we could just guess a solution. Instead we try to find a particular solution using the method of variation of parameters. From above we have that two linearly independent solutions to the associated homogeneous equation are given by

$$y_1 = e^{3x} \quad \text{and} \quad xe^{3x}.$$

Following the method of variation of parameters the particular solution will be given by $y_p = u_1y_1 + u_2y_2$ with

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

and

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2(x) \\ \dfrac{e^{3x}}{1+x^2} & y_2'(x) \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \dfrac{e^{3x}}{1+x^2} \end{vmatrix}.$$

We start computing

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (e^{3x} + 3xe^{3x}) \end{vmatrix} = e^{3x}(e^{3x} + 3xe^{3x}) - xe^{3x}3e^{3x} = e^{6x},$$

$$W_1 = \begin{vmatrix} 0 & xe^{3x} \\ \dfrac{e^{3x}}{1+x^2} & (e^{3x} + 3xe^{3x}) \end{vmatrix} = -\frac{e^{3x}}{1+x^2}xe^{3x} = -\frac{xe^{6x}}{1+x^2},$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \dfrac{e^{3x}}{1+x^2} \end{vmatrix} = e^{3x}\frac{e^{3x}}{1+x^2} = \frac{e^{6x}}{1+x^2}.$$

This gives us

$$u_1' = \frac{W_1}{W} = -\frac{\frac{xe^{6x}}{1+x^2}}{e^{6x}} = -\frac{x}{1+x^2}, u_2' = \frac{W_2}{W} = \frac{\frac{e^{6x}}{1+x^2}}{e^{6x}} = \frac{1}{1+x^2}.$$

The change of variables $u = x^2$ gives us $u_1 = -\dfrac{1}{2}\log(1+x^2)$ and for u_2 we immediately get $u_2 = \tan^{-1} x$. Notice that we skip the integration constants since we are only looking for one solution. This gives us the particular solution

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2}\log(1+x^2)e^{3x} + \tan^{-1}(x)xe^{3x}.$$

The general solution is then given by

$$\begin{aligned} y = y_h + y_p &= e^{3x}(C_1 + C_2 x) - \frac{1}{2}\log(1+x^2)e^{3x} + \tan^{-1}(x)xe^{3x} \\ &= e^{3x}(C_1 + C_2 x - \frac{1}{2}\log(1+x^2) + x\tan^{-1} x). \end{aligned}$$

Solution to problem 4. **a)** To show that $x = 0$ is a regular singular point we first notice that $p(x) = \dfrac{x^2+x}{x^2} = 1 + \dfrac{1}{x}$ and $q(x) = -\dfrac{2}{x^2}$, both of these are clearly singular at $x = 0$. To see that it's regular we check that $xp(x) = x + 1$ and $x^2 q(x) = -2$ are both analytic, which is clearly the case.

b) The indicial equation is given by $r^2 + (p_0 - 1)r + q_0 = 0$ where $p_0 = xp(0) = 1$ and $q_0 = x^2 q(0) = -2$. Giving us the indicial equation $r^2 - 2 = 0$, which has the roots $r_{1,2} = \pm\sqrt{2}$.

c) According to the method of Frobenious we are looking for a series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where $r = \sqrt{2}$ is the larger of the two roots of the indicial equation.

We have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ y' &= \sum_{n=0}^{\infty} (n+r)a_n x^{n-1+r}, \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n-1+r)a_n x^{n-2+r}. \end{aligned}$$

We want to plug this into the equation, to make the calculations slightly easier we first simplify each term. We notice that

$$\begin{aligned} x^2 y'' &= \sum_{n=0}^{\infty} (n+r)(n-1+r)a_n x^{n+r}, \\ (x^2+x)y' &= \sum_{n=0}^{\infty} (n+r)a_n x^{n+1+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r}, \\ -2y &= \sum_{n=0}^{\infty} -2a_n x^{n+r}. \end{aligned}$$

Three of the sums have x^{n+r} and one of them have x^{n+1+r}. We rewrite them to all have x^{n+r} and also start at the same index, this gives us

$$\begin{aligned} x^2 y'' &= r(r-1)a_0 x^r + \sum_{n=1}^{\infty} (n+r)(n-1+r)a_n x^{n+r}, \\ (x^2+x)y' &= ra_0 x^r + \sum_{n=1}^{\infty} (n-1+r)a_{n-1} x^{n+r} + \sum_{n=1}^{\infty} (n+r)a_n x^{n+r}, \\ 2y &= -2a_0 x^r + \sum_{n=1}^{\infty} -2a_n x^{n+r}. \end{aligned}$$

Summing the three expressions we arrive at

$$(r(r-1)+r-2)a_0x^r + \sum_{n=1}^{\infty}(((n+r)(n-1+r)+(n+r)-2)a_n + (n-1+r)a_{n-1})x^{n+r} = 0.$$

Simplifying it we get

$$(r^2-2)a_0x^r + x^r\sum_{n=1}^{\infty}(((n+r)^2-2)a_n + (n-1+r)a_{n-1})x^n = 0.$$

For $r=\sqrt{2}$ the first term is zero, for the sum to equal zero we must (by the identity principle) have

$$((n+r)^2-2)a_n + (n-1+r)a_{n-1} = 0 \iff a_n = -\frac{n-1+r}{(n+r)^2-2}a_{n-1}$$

for $n = 1, 2, \ldots$. Since we are only looking for a particular solution we can take $a_0 = 1$ and this together with $r = \sqrt{2}$ gives us

$$a_n = -\frac{n-1+\sqrt{2}}{(n+\sqrt{2})^2-2}a_{n-1} = -\frac{n-1+\sqrt{2}}{n^2+2\sqrt{2}n}a_{n-1}$$

and the three first terms are given by

$$\begin{aligned}
a_0 &= 1,\\
a_1 &= -\frac{\sqrt{2}}{1+2\sqrt{2}}a_0 = -\frac{\sqrt{2}}{1+2\sqrt{2}},\\
a_2 &= -\frac{1+\sqrt{2}}{4+4\sqrt{2}}a_1 = \frac{1+\sqrt{2}}{4+4\sqrt{2}}\frac{\sqrt{2}}{1+2\sqrt{2}} = \frac{\sqrt{2}+2}{20+12\sqrt{2}}.
\end{aligned}$$

Solution to problem 5. We have a homogeneous linear system with constant coefficients, we can hence write it as

$$X' = AX \text{ with } X' = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}.$$

To find the general solution we need to compute the eigenvalues and eigenvectors of A. For the eigenvalues we get

$$\begin{vmatrix} 3-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - 5 = \lambda^2 - 2\lambda - 8,$$

which has the zeros $\lambda_1 = -2$ and $\lambda_2 = 4$.

Next step is to compute the eigenvectors. For $\lambda_1 = -2$ we get

$$\begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for which one solution is $k_1 = -k_2 = 1$. Giving us the first eigenvector

$$K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 4$ we get

$$\begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives us (for example)

$$K_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Combining the eigenvalues and the eigenvectors we find that the general solution is given by

$$X = C_1K_1e^{\lambda_1} + C_2K_2e^{\lambda_2} = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 5 \\ 1 \end{pmatrix}e^{4t},$$

or equivalently

$$\begin{cases} x &= C_1e^{-2t} + 5C_2e^{4t} \\ y &= -C_1e^{-2t} + C_2e^{4t} \end{cases}.$$

The phase portrait is a saddle point with K_1 giving the stable direction and K_2 the unstable direction.

Solution to problem 6. **a)** The system is on the form

$$X' = AX + F(t)$$

with

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} \dfrac{e^t}{1-e^{2t}} \\ 0 \end{pmatrix}.$$

The existence and uniqueness theorem tells us that there exists a unique solution to the given initial value problem on any interval I for which $F(t)$ is continuous. We can note that $F(t)$ is continuous as long as $1 - e^{2t} \neq 0$, so as long as $x \neq 0$. A solution is therefore defined on any interval I which contains t_0 but not 0.

b) We begin by solving the associated homogeneous system $X' = AX$, with X and A as above. Since A is upper triangular we can read the eigenvalues from the diagonal, they are given by $\lambda_1 = -1$ and $\lambda_2 = 1$. For the eigenvectors we get for $\lambda_1 = -1$

$$\begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for which one solution is $k_1 = 1$, $k_2 = 0$. Giving us the first eigenvector

$$K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 1$ we get

$$\begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives us (for example)

$$K_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So the general solution to the homogeneous equation is

$$X_h = C_1 K_1 e^{\lambda_1 t} + C_2 K_2 e^{\lambda_2 t} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

and in particular

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \text{ and } X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

form a fundamental set of solutions.

Next we want to find a particular solution. The right hand side is not on any simple form so we will use the method of variation of parameters for finding one. For this we need the fundamental matrix of the system, which in this case is given by

$$\Phi = \begin{pmatrix} e^{-t} & e^t \\ 0 & -e^t \end{pmatrix}.$$

According to the method of variation of parameters a particular solution is then given by

$$X_p = \Phi(t) \int \Phi^{-1}(t) F(t)\, dt.$$

We have that $\det \Phi = e^{-t}(-e^t) = -1$ so

$$\Phi^{-1}(t) = \frac{1}{\det \Phi} \begin{pmatrix} -e^t & -e^t \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & e^t \\ 0 & -e^{-t} \end{pmatrix}.$$

This gives us

$$X_p = \Phi(t) \int \begin{pmatrix} e^t & e^t \\ 0 & -e^{-t} \end{pmatrix} \begin{pmatrix} \dfrac{e^t}{1-e^{2t}} \\ 0 \end{pmatrix} dt = \Phi(t) \int \begin{pmatrix} \dfrac{e^{2t}}{1-e^{2t}} \\ 0 \end{pmatrix} dt = \Phi(t) \begin{pmatrix} \displaystyle\int \dfrac{e^{2t}}{1-e^{2t}}\, dt \\ 0 \end{pmatrix}.$$

We get that

$$\int \frac{e^{2t}}{1-e^{2t}}\, dt = -\frac{1}{2}\log|1-e^{2t}|,$$

where we ignore the constant of integration since we are only looking for one solution, and thus

$$X_p = \Phi(t)\begin{pmatrix} -\frac{1}{2}\log|1-e^{2t}| \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t} & e^t \\ 0 & -e^t \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\log|1-e^{2t}| \\ 0 \end{pmatrix} = -\frac{1}{2}e^{-t}\log|1-e^{2t}|\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So the general solution is given by

$$X = X_h + X_p = C_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t + -\frac{1}{2}e^{-t}\log|1-e^{2t}|\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution to problem 7. **a)** To rewrite the equation as a system we introduce the variables $x = u$ and $y = u'$. This gives us $x' = u' = y$ and also $y' = u'' = \mu(1-u^2)u' - u = \mu(1-x^2)y - x$, hence we have the system of first order equations

$$\begin{cases} x' &= y \\ y' &= \mu(1-x^2)y - x \end{cases}.$$

b) To find the critical points we notice that $x' = y$ immediately gives us $y = 0$. The second equation we need $\mu(1-x^2)y - x = 0$, but $y = 0$ reduces it to $-x = 0$ which means that we also have $x = 0$. We conclude that the origin is the only critical point of the system.

c) To determine the type and stability of the origin we will look at the linear part of the system. Rewriting the system as

$$\begin{cases} x' &= y \\ y' &= -x + \mu y - \mu x^2 y \end{cases}$$

we see that the linear part is given by

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

To determine the type and stability of the origin we are interested in the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

We get

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \mu-\lambda \end{vmatrix} = \lambda^2 - \mu\lambda + 1$$

which has the roots

$$\lambda_{1,2} = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - 1} = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right).$$

The type and stability of the origin depends on the eigenvalues and hence on μ. We get a few different cases to consider

(a) If $|\mu| < 2$ then $\mu^2 - 4 < 0$ so we get complex eigenvalues and the real part is given by $\mu/2$. If $-2 < \mu < 0$, so $\mu/2$ is negative, then we get a stable spiral. If instead $0 < \mu < 2$, so that $\mu/2 > 0$, we get an unstable spiral. The case $\mu = 0$ we are not asked about but we can note that in this case the real part is zero so linear part doesn't given any information about the type nor the stability.

(b) If $\mu = \pm 2$ we have $\mu^2 - 4 = 0$ so we get a double root $\lambda_{1,2} = \pm 1$. We don't learn anything about the type of the critical point in this case, but this is also not part of the question. For the stability we can note that $\mu = 2$ makes it unstable and $\mu = -2$ makes it stable.

(c) If $|\mu| > 0$ then $\mu^2 - 4 > 0$ so we have two distinct real roots λ_1 and λ_2. Since $\sqrt{\mu^2-4} < \sqrt{\mu^2} = |\mu|$ we find that if $\mu < 2$ then both λ_1 and λ_2 are negative and if $\mu > 2$ then both of them are positive. In the first case we get a stable node and in the second case an unstable node.

Solution to problem 8. **a)** The critical points are given by the solutions to the system

$$\begin{cases} 0 &= 2x^3 + 2y^3 \\ 0 &= x^3 - e^x y^7 \end{cases}.$$

The first equation gives us that $x^3 = -y^3 \iff x = -y$. Plugging $y = -x$ into the second equation we get

$$0 = x^3 + e^x x^7 \iff 0 = x^3(1 + e^x x^4).$$

Since $1 + e^x x^4$ is always positive the only way for this to be zero is if $x = 0$, in which case $y = 0$ as well. We conclude that the origin is the *only* critical point of the system, since it's the only critical point it is also isolated.

b) To begin with we look for a, b and k, l such that $\frac{dV}{dt}$ has a definite sign. We find that

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \\ &= kax^{k-1}(2x^3 + 2y^3) + lby^{l-1}(x^3 - e^x y^7) \\ &= 2kax^{k+2} + 2kax^{k-1}y^3 + lbx^3y^{l-1} - lbe^x y^{l+6}. \end{aligned}$$

The terms $2kax^{k+2}$ and $-lbe^x y^{l+6}$ both have definite signs (depending on the sign of a and b). The two other terms have varying signs so we try to make them cancel out. For that we need

$$2kax^{k-1}y^3 + lbx^3y^{l-1} = 0 \iff 2kax^{k-1}y^3 = -lbx^3y^{l-1},$$

for all x, y, which only holds if $k - 1 = 3$, $3 = l - 1$ and $2ka = -lb$. We get $k = 4$, $l = 4$ and then $8a = -4b$ so we can take for example $a = 1$ and $b = -2$. This gives us

$$\frac{dV}{dt} = 8x^6 + 8e^x y^{10},$$

which is positive definite.

Since $\frac{dV}{dt}$ is positive definite it's enough to show that $V(x, y) = x^4 - 2y^4$ is positive at some point in every neighbourhood of the origin to get that the point is unstable. Clearly V is positive whenever $y = 0$ and hence it's positive at some point in every neighbourhood of the origin as well.