UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
KandFy3, gylärare,STS, frist.
1MA032 Introduction to ODEs
2022-04-12

Writing time: 5 hrs. Allowed accessories: writing materials only. The credit for each problem is shown below. For the grades 3, 4 and 5, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be accompanied with explanatory text. Maximum one solution per page.

1. (5 points) Solve the initial value problem

$$xy' + 2x^3 = 3x^4 + y \quad y(1) = 1 \quad \text{with} \quad x > 0.$$

On which interval is the solution defined?

2. (5 points) The function $y_1(x) = \dfrac{1}{x^2}$ is a solution of

$$x^2y'' + xy' - 4y = 0, \quad x > 0.$$

Solve the corresponding initial value problem with $y(1) = 0$ and $y'(1) = 1$.

3. (5 points)

a) Give the general solution to the ODE

$$y'' - 5y' + 6y = \sin(x) + 2x$$

b) Given an example of a second order linear ODE with constant coefficients that has

$$y_1(x) = 2e^{2x} + e^{3x} + x^3e^{5x}$$

and

$$y_2(x) = e^{2x} + 2e^{3x} + x^3e^{5x}$$

as solutions.

4. (5 points) Consider the differential equation

$$y'' - \left(\frac{1}{2x} + 1\right) y' + \frac{1}{2x^2} y = 0$$

a) Show that this equation has a regular singular point at $x = 0$.
b) Determine the indicial equation and its roots.
c) Find two series solutions for $x > 0$, one corresponding to each of the roots of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients.

5. (5 points) Consider the inhomogeneous system

$$\begin{cases} x' &= x - 2y + e^t \\ y' &= x + 4y + e^{-t}. \end{cases}$$

Use the method of variation of parameters to find the general solution.

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6. (5 points) Consider the ODE

$$u'' + \sin(u') - e^{5u'}(u')^2 + 2u = 0$$

a) Reduce the ODE to a system of first order equations.

b) Find all critical points and classify their type and stability.

c) Are there periodic trajectories contained strictly in the upper half-plane?

7. (5 points) Consider the linear system

$$\begin{cases} x' &= 2x + y \\ y' &= \alpha x + y, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq 2$. Investigate the type and stability of the origin $(0,0)$ depending on α. Sketch the phase portrait for $\alpha = 0$.

8. (5 points) Consider the non-linear system

$$\begin{cases} x' &= 2xy + x \\ y' &= x^{30} - 2y^5. \end{cases}$$

a) Find the critical points of the system.

b) Use Lyapunov's method, with a function of the form $V(x,y) = ax^k + cy^\ell$, to determine the type of stability of the origin $(0,0)$.

GOOD LUCK!

Svar till tentamen i 1MA032 Introduction to ODEs 2022-04-12