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Prov i matematik
KandFy3, KandKe3, gyläre, STS, frist.
1MA032 Introduction to ODEs
2022-04-12

Writing time: 5 hrs. Allowed accessories: writing materials only. The credit for each problem is shown below. For the grades 3, 4 and 5, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be accompanied with explanatory text. Maximum one solution per page.

1. (5 points) Solve the initial value problem

$$xy' + 2x^3 = 3x^4 + y \quad y(1) = 1 \quad \text{with} \quad x > 0.$$

On which interval is the solution defined?

Suggested solution: By rewriting the equation as

$$y' - \frac{1}{x}y = 3x^3 - 2x^2$$

we see that it is a first order linear ODE. The integrating factor is given by

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\log x} = \frac{1}{x}.$$

Multiplying the equation by $\mu(x)$ we get

$$\frac{1}{x}y' - \frac{1}{x^2}y = 3x^2 - 2x,$$

where the left hand side can be written as $\left(\frac{y}{x}\right)'$, giving us

$$\left(\frac{y}{x}\right)' = 3x^2 - 2x.$$

Integrating both sides we have

$$\frac{y}{x} = \int 3x^2 - 2x\,dx = x^3 - x^2 + C.$$

Solving for y gives us

$$y = x^4 - x^3 + Cx.$$

Using the the initial condition $y(1) = 1$ we have

$$1 = 1^4 - 1^3 + C \cdot 1 \iff C = 1.$$

The solution to the initial value problem is hence given by

$$y(x) = x^4 - x^3 + x,$$

which is defined on the whole real line.

2. (5 points) The function $y_1(x) = \frac{1}{x^2}$ is a solution of

$$x^2y'' + xy' - 4y = 0, \quad x > 0.$$

Solve the corresponding initial value problem with $y(1) = 0$ and $y'(1) = 1$.

Suggested solution: We start by using the method of reduction of order to find the general solution to the ODE. We let $y_2 = y_1 g$ and insert it into the ODE, giving us

$$x^2(y_1 g)'' + x(y_1 g)' - 4y_1 g = 0.$$

We have

$$\begin{aligned}(y_1 g)' &= y_1' g + y_1 g',\\ (y_1 g)'' &= y_1'' g + 2y_1' g' + y_1 g''.\end{aligned}$$

Inserting this into the equation we have

$$x^2(y_1'' g + 2y_1' g' + y_1 g'') + x(y_1' g + y_1 g') - 4y_1 g = 0.$$

Reordering the terms we can write this as

$$g(x^2 y_1'' + x y_1' - 4y_1) + 2x^2 y_1' g' + x^2 y_1 g'' + x y_1 g' = 0,$$

where the first term is zero since y_1 is a solution of the ODE. We are left with

$$x^2 y_1 g'' + (2x^2 y_1' + x y_1) g' = 0.$$

If we let $h = g'$ we can write this as

$$x^2 y_1 h' + (2x^2 y_1' + x y_1) h = 0.$$

Inserting

$$y_1(x) = \frac{1}{x^2}, \quad y_1'(x) = -\frac{2}{x^3}$$

gives us

$$h' - \frac{3}{x} h = 0.$$

This equation is linear and an integrating factor is given by $\mu(x) = x^{-3}$, giving us

$$(x^{-3} h)' = 0$$

and hence

$$h = Cx^3.$$

Since $g' = h$ this gives us $g = \frac{C}{4} x^4 + D$. We get

$$y_2(x) = y_1(x) g(x) = \frac{C}{4} x^2 + \frac{D}{x^2}.$$

Taking $C = 4$ and $D = 0$ gives us $y_2(x) = x^2$ as a second solution. The general solution is hence given by

$$y(x) = \frac{C}{x^2} + Dx^2.$$

We have

$$y'(x) = -\frac{2C}{x^3} + 2Dx.$$

For the initial conditions we have

$$y(1) = C + D \quad \text{and} \quad u'(1) = -2C + 2D.$$

so the initial condition $y(1) = 0$ and $y'(1) = 1$ gives us the linear system

$$\begin{aligned}0 &= C + D,\\ 1 &= -2C + 2D.\end{aligned}$$

The solutions is given by $C = -\frac{1}{4}$ and $D = \frac{1}{4}$ and we get the solution

$$y(x) = -\frac{1}{4x^2} + \frac{x^2}{4}$$

3. (5 points)

a) Give the general solution to the ODE

$$y'' - 5y' + 6y = \sin(x) + 2x$$

b) Given an example of a second order linear ODE with constant coefficients that has

$$y_1(x) = 2e^{2x} + e^{3x} + x^3e^{5x}$$

and

$$y_2(x) = e^{2x} + 2e^{3x} + x^3e^{5x}$$

as solutions.

Suggested solution: We start by finding the general solution to the associated homogeneous system $y'' - 5y' + 6y = 0$. The characteristic equation is given by $r^2 - 5r + 6 = 0$, which has the roots $r_1 = 2$, $r_2 = 3$. The general solution to the homogeneous equation is thus given by

$$y_h = C_1e^{2x} + C_2e^{3x}.$$

To find a particular solution we split the right hand side into two parts, $\sin(x)$ and $2x$. For $\sin(x)$ a natural guess for the particular solution would be $y_{p,1}(x) = A\sin(x) + B\cos(x)$, this gives us

$$\begin{aligned} y'_{p,1}(x) &= A\cos(x) - B\sin(x), \\ y''_{p,1}(x) &= -A\sin(x) - B\cos(x). \end{aligned}$$

Inserting it into the equation we have

$$\begin{aligned} y''_{p,1}(x) - 5y'_{p,1}(x) + 6y_{p,1}(x) &= -A\sin(x) - B\cos(x) - 5A\cos(x) + 5B\sin(x) + 6A\sin(x) + 6B\cos(x) \\ &= (5A + 5B)\sin(x) + (-5A + 5B)\cos(x). \end{aligned}$$

For this to equal $\sin(x)$ we must have $5A + 5B = 1$ and $-5A + 5B = 0$, giving us $A = B = \frac{1}{10}$. For $2x$ a natural guess would be $y_{p,2}(x) = Cx + D$, giving us

$$\begin{aligned} y'_{p,2}(x) &= C, \\ y''_{p,2}(x) &= 0. \end{aligned}$$

Inserting it into the equation we have

$$y''_{p,2}(x) - 5y'_{p,2}(x) + 6y_{p,2}(x) = 0 - 5C + 6Cx + 6D = 6Cx + (-5C + 6D).$$

For this to equal $2x$ we must have $6C = 2$ and $-5C + 6D = 0$, giving us $C = \frac{1}{3}$ and $D = \frac{5}{18}$. A particular solution to the full equation is hence given by

$$y_p(x) = y_{p,1}(x) + y_{p,2}(x) = \frac{\sin(x) + \cos(x)}{10} + \frac{x}{3} + \frac{5}{18}.$$

The general solution to the full equation is then given by

$$y(x) = y_h(x) + y_p(x) = C_1e^{2x} + C_2e^{3x} + \frac{\sin(x) + \cos(x)}{10} + \frac{x}{3} + \frac{5}{18},$$

which solves *a*).

To solve *b*) we have from the previous problem that $C_1e^{2x} + C_2e^{3x}$ is a solution to the homogeneous second order linear ODE

$$y'' - 5y' + 6y = 0$$

for any choice of C_1 and C_2. This means that the term $2e^{2x} + e^{3x}$ in y_1 and the term $e^{2x} + 2e^{3x}$ in y_2 would disappear if we insert them into the equation. In particular we get that

$$y_1'' - 5y_1' + 6y_1 = y_2'' - 5y_2' + 6y_2 = (x^3e^{5x})'' - 5(x^3e^{5x})' + 6x^3e^{5x}.$$

If we let

$$f(x) = (x^3e^{5x})'' - 5(x^3e^{5x})' + 6x^3e^{5x}$$

both y_1 and y_2 would then solve the equation

$$y'' - 5y' + 6y = f(x).$$

Computing $f(x)$ we first note that

$$\begin{aligned}(x^3e^{5x})' &= 3x^2e^{5x} + 5x^3e^{5x} = (3x^2 + 5x^3)e^{5x},\\ (x^3e^{5x})'' &= (6x + 15x^2)e^{5x} + 5(3x^2 + 5x^3)e^{5x} = (6x + 30x^2 + 25x^3)e^{5x}.\end{aligned}$$

This gives us

$$f(x) = (6x + 30x^2 + 25x^3)e^{5x} - 5(3x^2 + 5x^3)e^{5x} + 6x^3e^{5x} = (6x + 15x^2 + 6x^3)e^{5x}.$$

So y_1 and y_2 solves the ODE

$$y'' - 5y' + 6y = (6x + 15x^2 + 6x^3)e^{5x}.$$

4. (5 points)Consider the differential equation

$$y'' - \left(\frac{1}{2x} + 1\right)y' + \frac{1}{2x^2}y = 0$$

a) Show that this equation has a regular singular point at $x = 0$.

b) Determine the indicial equation and its roots.

c) Find two series solutions for $x > 0$, one corresponding to each of the roots of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients.

Suggested solution: To show that $x = 0$ is a regular singular point we first note that the equation can be written as

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x) = -\left(\frac{1}{2x} + 1\right)$ and $q(x) = \frac{1}{2x^2}$. Both $p(x)$ and $q(x)$ are clearly singular at $x = 0$, so $x = 0$ is a singular point. To see that it's regular we check that $xp(x) = -\left(\frac{1}{2} + x\right)$ and $x^2q(x) = \frac{1}{2}$ are both analytic, which is clearly the case.

The indicial equation is given by $r^2 + (p_0 - 1)r + q_0 = 0$ where p_0 is the value of $xp(x)$ at $x = 0$ and q_0 is the value of $x^2q(x)$ at $x = 0$. We get $p_0 = -\frac{1}{2}$ and $q_0 = \frac{1}{2}$. Giving us the indicial equation $r^2 - \frac{3}{2}r + \frac{1}{2} = 0$, which has the roots $r_1 = \frac{1}{2}$, $r_2 = 1$.

According to the method of Frobenious we are looking for series solutions of the form

$$y = x^r\sum_{n=0}^{\infty} a_nx^n$$

where r is one of the roots of the indicial equation.

We have

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\
y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n-1+r}, \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n-1+r) a_n x^{n-2+r}.
\end{aligned}
$$

We want to plug this into the equation, we get the three terms

$$
\begin{aligned}
y'' &= \sum_{n=0}^{\infty} (n+r)(n-1+r) a_n x^{n-2+r}, \\
-\left(\frac{1}{2x}+1\right) y' &= -\sum_{n=0}^{\infty} \frac{(n+r)}{2} a_n x^{n-2+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n-1+r}, \\
\frac{1}{2x^2} y &= \sum_{n=0}^{\infty} \frac{1}{2} a_n x^{n-2+r}.
\end{aligned}
$$

Three of the sums have x^{n-2+r} and one have x^{n-1+r}. We rewrite them all to have x^{n-2+r} and also start at the same index, this gives us

$$
\begin{aligned}
y'' &= r(r-1) a_0 x^{-2+r} + \sum_{n=1}^{\infty} (n+r)(n-1+r) a_n x^{n-2+r}, \\
-\left(\frac{1}{2x}+1\right) y' &= -\frac{r}{2} a_1 x^{-2+r} - \sum_{n=1}^{\infty} \frac{(n+r)}{2} a_n x^{n-2+r} - \sum_{n=1}^{\infty} (n-1+r) a_{n-1} x^{n-2+r}, \\
\frac{1}{2x^2} y &= \frac{1}{2} a_0 x^{-2+r} + \sum_{n=1}^{\infty} \frac{1}{2} a_n x^{n-2+r}.
\end{aligned}
$$

Summing the three expressions we arrive at

$$
\begin{aligned}
r(r-1) a_0 x^{-2+r} &- \frac{1+r}{2} a_1 x^{-2+r} + \frac{1}{2} a_0 x^{-2+r} \\
&+ \sum_{n=1}^{\infty} \left(\left((n+r)(n-1+r) - \frac{n+r}{2} + \frac{1}{2} \right) a_n - (n-1+r) a_{n-1} \right) x^{n-2+r} = 0.
\end{aligned}
$$

Simplifying it we get

$$
\left(r^2 - \frac{3}{2} r + \frac{1}{2} \right) a_0 x^{r-2} + x^{r-2} \sum_{n=1}^{\infty} \left(\left(n + r - \frac{1}{2} \right)(n+r-1) a_n - (n+r-1) a_{n-1} \right) x^n = 0.
$$

When r is a root of the indicial equation the first term is zero, for the sum to equal zero we must (by the identity principle) have

$$
\left(n + r - \frac{1}{2} \right)(n+r-1) a_n - (n+r-1) a_{n-1} = 0 \iff a_n = \frac{1}{n + r - \frac{1}{2}} a_{n-1}
$$

for $n = 1, 2, \ldots$. Since we are only looking for a particular solution we can take $a_0 = 1$.
For $r_1 = \frac{1}{2}$ we get

$$
a_n = \frac{1}{n} a_{n-1}
$$

and the three first terms are given by

$$
\begin{aligned}
a_0 &= 1, \\
a_1 &= \frac{1}{1} a_0 = 1, \\
a_2 &= \frac{1}{2} a_1 = \frac{1}{2}.
\end{aligned}
$$

For $r_2 = 1$ we get

$$a_n = \frac{1}{n + \frac{1}{2}} a_{n-1}$$

and the three first terms are given by

$$\begin{aligned} a_0 &= 1, \\ a_1 &= \frac{1}{1 + \frac{1}{2}} a_0 = \frac{2}{3}, \\ a_2 &= \frac{1}{2 + \frac{1}{2}} a_1 = \frac{4}{15}. \end{aligned}$$

5. (5 points) Consider the inhomogeneous system

$$\begin{cases} x' &= x - 2y + e^t \\ y' &= x + 4y + e^{-t}. \end{cases}$$

Use the method of variation of parameters to find the general solution.

Suggested solution: We write the equation as $X' = AX + F$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \qquad \text{and} \qquad F(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}. \tag{1}$$

It is not difficult to find the eigenvalues of A (via $0 = \det(A - \lambda I)$) to be $\lambda_+ = 3$ and $\lambda_- = 2$ with eigenvectors

$$X_- = e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \text{and} \qquad X_+ = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{2}$$

The fundamental matrix is

$$\Phi(t) = \begin{pmatrix} 2e^{2t} & e^{3t} \\ -e^{2t} & -e^{3t} \end{pmatrix}. \tag{3}$$

The particular solution is now given by

$$X_p(t) = \Phi(t) \int \Phi^{-1}(s) F(s) ds. \tag{4}$$

The inverse fundamental matrix is

$$\Phi^{-1}(s) = \begin{pmatrix} e^{-2t} & e^{-2t} \\ -e^{-3t} & -2e^{-3t} \end{pmatrix}. \tag{5}$$

Thus

$$\begin{aligned} X_p(t) &= \Phi(t) \int \Phi^{-1}(s) F(s) ds \\ &= \begin{pmatrix} 2e^{2t} & e^{3t} \\ -e^{2t} & -e^{3t} \end{pmatrix} \int \begin{pmatrix} e^{-2s} & e^{-2s} \\ -e^{-3s} & -2e^{-3s} \end{pmatrix} \begin{pmatrix} e^s \\ e^{-s} \end{pmatrix} ds \\ &= \begin{pmatrix} 2e^{2t} & e^{3t} \\ -e^{2t} & -e^{3t} \end{pmatrix} \int \begin{pmatrix} e^{-s} + e^{-3s} \\ -e^{-2s} - 2e^{-4s} \end{pmatrix} ds \\ &= \begin{pmatrix} 2e^{2t} & e^{3t} \\ -e^{2t} & -e^{3t} \end{pmatrix} \begin{pmatrix} -e^{-t} - \frac{e^{-3t}}{3} \\ \frac{e^{-2t}}{2} + \frac{e^{-4t}}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3e^t}{2} - \frac{e^{-t}}{6} \\ \frac{e^t}{2} - \frac{e^{-t}}{6} \end{pmatrix}. \end{aligned} \tag{6}$$

Hence, the total solution is

$$X(t) = X_h(t) + X_p(t) = e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{e^t}{2} \begin{pmatrix} -3 \\ 1 \end{pmatrix} - \frac{e^{-t}}{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7}$$

6. (5 points) Consider the ODE

$$u'' + \sin(u') - e^{5u'}(u')^2 + 2u = 0.$$

a) Reduce the ODE to a system of first order equations.

b) Find all critical points and classify their type and stability.

c) Are there periodic trajectories contained strictly in the upper half-plane?

Suggested solution: By invoking the variables $x = u$ and $y = u'$ we find our ODE reduces to

$$\begin{cases} x' &= y, \\ y' &= -2x - \sin(y) + e^{5y}y^2. \end{cases} \tag{8}$$

It is immediate that if both $x' = 0$ and $y' = 0$ we find $x = y = 0$. Writing $x' = P(x, y)$ and $y' = Q(x, y)$ we find that both P and Q are analytic at $(x, y) = (0, 0)$. In particular, $Q(x, y) = -2x - y - (\sin(y) - y) + e^{5y}y^2$ and it is easily seen that

$$\lim_{(x,y)\to(0,0)} \frac{-(\sin(y) - y) + e^{5y}y^2}{\sqrt{x^2 + y^2}} = 0, \tag{9}$$

via, for instance, a change to polar coordinates after a McLaurin expansion $\sin(y) = y + O(y^3)$. Hence the system is locally linear. The linear part of the system reads in matrix form $X' = AX$, where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \tag{10}$$

and it is straightforward to find the eigenvalues to be

$$\lambda_{\pm} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}, \tag{11}$$

so that the origin is a **stable spiral**. Moreover, since the origin is the only critical point any orbit must enclose it. Hence there cannot be any closed orbit in the upper halfplane.

Please turn the page!

7. (5 points) Consider the linear system

$$\begin{cases} x' &= 2x + y \\ y' &= \alpha x + y, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq 2$. Investigate the type and stability of the origin $(0, 0)$ depending on α. Sketch the phase portrait for $\alpha = 0$.

Suggested solution: It is not difficult to see (via the calculation $0 = \det(A - \lambda I)$) that the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ \alpha & 1 \end{pmatrix} \tag{12}$$

has eigenvalues

$$\lambda_\pm(\alpha) = \frac{3}{2} \pm \sqrt{\frac{1}{4} + \alpha}. \tag{13}$$

Hence, we see that when $\frac{1}{4} + \alpha \geq 0$ the eigenvalues are real (and non-zero since $\alpha \neq 2$), and othewise complex with positive real part. In the former case the origin is a saddle point and in the latter case the origin is an unstable spiral. In the special case of $\alpha = 0$ we find $\lambda_+(0) = 2$ and $\lambda_-(0) = 1$ and the origin is a saddlepoint. The associated eigenvectors are

$$X_-(\alpha = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad X_+(\alpha = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{14}$$

The phase portrait for $\alpha = 0$ is drawn in Figure 1. Far out along the trajectories (i.e. for large $t >> 0$) the factor e^{2t} dominates and so the trajectories asymptote towards the X_+-line. For $t << 0$ the factor e^t dominates and so the trajectories converge to the origin along the X_--line.

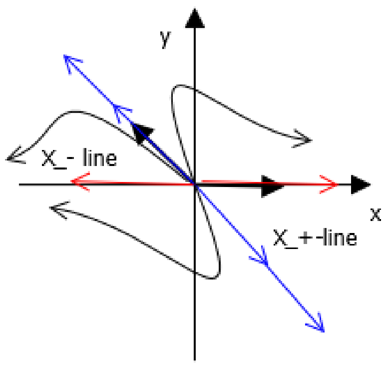

Figure 1: Phase portrait with the X_--line in blue and the X_+-line in red.

8. (5 points) Consider the non-linear system

$$\begin{cases} x' &= 2xy + x \\ y' &= x^{30} - 2y^5. \end{cases}$$

a) Find the critical points of the system.

b) Use Lyapunov's method, with a function of the form $V(x, y) = ax^k + cy^\ell$, to determine the type of stability of the origin $(0, 0)$.

Suggested solution: To solve for critical points we equate $x' = y' = 0$:

$$\begin{cases} 0 &= 2xy + x = x(2y + 1), \\ 0 &= x^{30} - 2y^5. \end{cases}$$

If $x = 0$ then $y = 0$ follows. If $x \neq 0$, then the first equation forces $y = -\frac{1}{2}$, whereafter the second equation forces $x^{30} + 2^{-4} = 0$ which does not admit real solutions. Hence the origin is the only critical point. To determine the stability of the origin we attempt Lyapunov's method, with V as suggested:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \\ &= akx^{k-1}(2xy + x) + c\ell y^{\ell-1}(x^{30} - 2y^5) \\ &= 2akx^k y + akx^k + c\ell y^{\ell-1}x^{30} - 2c\ell y^{\ell+4}.\end{aligned}$$

We see that for $k = 30$ and $\ell = 2$ the terms mixed in x and y will have the same powers, and so

$$\frac{dV}{dt} = a30x^{30} - 4cy^6 + 2(30a + c)x^{30}y.$$

Choosing, for instance, $a = 1$ and $c = -30$ we find that

$$V(x, y) = x^{30} - 30y^2 \tag{15}$$

has positive derivative: $\frac{dV}{dt} > 0$ outside of $(0, 0)$. Furthermore $V(x, y)$ is positive along the x-axis. Hence the origin is **unstable** from the Lyapunov theory discussed in class.

GOOD LUCK!

Svar till tentamen i 1MA032 Introduction to ODEs 2022-04-12