UPPSALA UNIVERSITET
Matematiska institutionen
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Tentamen i Reell Analys 10p
1MA226
Kand Mat mfl

Duration: 08:00–13:00. The exam consists of 8 problems, each worth 5 points. All solutions should be provided with details and appropriate justifications. No calculators are allowed.

1. Show that $\inf_{1<x<2} \frac{1+2x}{x} = \frac{5}{2}$.
2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:
   (a) $x_n = (-1)^n n$.
   (b) $x_n = (1+\frac{1}{n})^n(-1)^n + \sin\frac{n\pi}{4}$.
3. Show that the sequence $a_n := \int_\pi^{n\pi} \frac{\sin x}{x}\, dx$ converges in $\mathbb{R}$. Hint: Use integration by parts.
4. Show that the function $F(x) = \sum_{n=1}^\infty e^{-nx}\cos n\pi x$ is differentiable in the interval $[0,\infty)$. Thereafter calculate the exact numerical value of $F'(1)$.
5. Show that if a continuous function $f : [0,1] \mapsto \mathbb{R}$ satisfies $\int_0^1 f(x)x^{(\frac{1}{2n+1})}\, dx = 0$ for $n = 0, 1, 2, \dots,$ then $f(x) = 0$ for all x in $[0,1]$. Does this statement hold true if the interval $[0,1]$ is replaced by $[-1,1]$? In order to obtain full credit, you need to fully justify all the steps of your solution.
6. Assume that $c_0,\ c_1,\ \cdots,\ c_n$ are real numbers so that $\sum_{k=0}^n \frac{c_k}{k+1} = 0$. Prove that the polynomial $p(x) = \sum_{k=0}^n c_k x^k$ has a zero in $[0,1]$. (This means that there exists a point $\xi \in [0,1]$ such that $p(\xi) = 0$).
7. Let $K(x,y) \in C([0,1]\times[0,1])$ and assume that $|K(x,y)| \le \frac{1}{2}$ for all $(x,y) \in [0,1]\times[0,1]$. Show that there exists a unique solution $f(x)$ to the integral equation,
$$f(x) = \int_0^1 K(x,y)f(y)\, dy.$$
8. The system
$$\begin{cases} u+v+w=6 \\ u^2+v^2+w^2=14 \end{cases}$$
is satisfied at the point $(1,2,3)$. Show that u and v can be solved in a neighbourhood of $(1,2,3)$ as a function of w. Calculate also $u'(3)$ and $v'(3)$, where u and v are regarded as functions of w.

Comments:

Problem 4: Typo; "$[0,\infty)$" should be "$(0,\infty)$".

Problem 5: I think that there is a typo and that "$\int_0^1 f(x)x^{(\frac{1}{2n+1})}\,dx = 0$" should be corrected to "$\int_0^1 f(x)x^{2n+1}\,dx = 0$", otherwise I do not see at present how to solve the first part of the problem.

Problem 7: One could make the problem a bit more precise by writing "... a unique solution $f(x)$ *in* $C([0,1])$ to the integral equation ...".

— A.S.