Exam, Real analysis, 1MA226, 2014-04-23

Solutions.

1. The task is to prove that $\frac{5}{2} = \inf E$ where $E = \{\frac{1+2x}{x} : 1 < x < 2\}$.

Note that for every $1 < x < 2$ we have $\frac{1+2x}{x} = \frac{1}{x} + 2 > \frac{1}{2} + 2 = \frac{5}{2}$. Hence $\frac{5}{2}$ is a lower bound of E.

Next assume that $\beta > \frac{5}{2}$. Then $\beta - 2 > \frac{1}{2}$ and thus $\frac{1}{\beta-2} < 2$; therefore there exists a real number x which satisfies $\max(1, \frac{1}{\beta-2}) < x < 2$. Then $1 < x < 2$ and also $\frac{1+2x}{x} = \frac{1}{x} + 2 < (\beta - 2) + 2 = \beta$, i.e. E contains a number which is $< \beta$. Hence β is not a lower bound of E. We have thus proved that no number $\beta > \frac{5}{2}$ is a lower bound of E.

Hence $\inf E = \frac{5}{2}$. $\square$

2. (a) Note that the subsequence $x_2, x_4, x_6, \dots$ tends to $+\infty$; hence $\limsup_{n\to\infty} x_n = +\infty$ (by Theorem 3.17). Similarly, the subsequence $x_1, x_3, x_5, \dots$ tends to $-\infty$; hence $\liminf_{n\to\infty} x_n = -\infty$.

Answer: $\limsup_{n\to\infty} x_n = +\infty$. $\liminf_{n\to\infty} x_n = -\infty$.

(b) We partition the sequence (x_n) into the 8 subsequences $(x_{j+8k})_{k=0,1,2,\dots}$, for $j = 1, 2, \dots, 8$. Note that each of these subsequences converges! Indeed, for $n = j + 8k$ we have $(-1)^n = (-1)^j$ and $\sin\left(\frac{n\pi}{4}\right) = \sin\left(\frac{j\pi}{4}\right)$; hence

$$\begin{aligned}
\lim_{k\to\infty} x_{j+8k} &= \lim_{k\to\infty}\left(\left(1+\frac{1}{j+8k}\right)^{j+8k}(-1)^{j+8k} + \sin\left(\frac{(j+8k)\pi}{4}\right)\right)\\
&= \lim_{k\to\infty}\left(\left(\left(1+\frac{1}{j+8k}\right)^{j+8k}(-1)^{j} + \sin\left(\frac{j\pi}{4}\right)\right)\\
&= (-1)^j e + \sin\left(\frac{j\pi}{4}\right)\\
&= \begin{cases} -e + 2^{-1/2} & \text{if } j = 1 \text{ or } 3\\ e + 1 & \text{if } j = 2\\ e & \text{if } j = 4 \text{ or } 8\\ -e - 2^{-1/2} & \text{if } j = 5 \text{ or } 7\\ e - 1 & \text{if } j = 6. \end{cases}
\end{aligned}$$

Hence the set of subsequential limits of (x_n) (in the extended real number system) is $\{-e + 2^{-1/2}, e+1, e, -e - 2^{-1/2}, e-1\}$, and so

$$\limsup_{n\to\infty} x_n = \sup\{-e + 2^{-1/2}, e+1, e, -e - 2^{-1/2}, e-1\} = e + 1$$

and

$$\liminf_{n\to\infty} x_n = \inf\{-e + 2^{-1/2}, e+1, e, -e - 2^{-1/2}, e-1\} = -e - 2^{-1/2}.$$

Answer: $\limsup_{n\to\infty} x_n = e + 1$ and $\liminf_{n\to\infty} x_n = -e - 2^{-1/2}$.

Alternative solution to (b), instead using Theorem 3.17: We claim that $\limsup_{n\to\infty} x_n = e+1$. To prove this, note that

$$\begin{aligned}\lim_{k\to\infty} x_{2+8k} &= \lim_{k\to\infty}\left(\left(1+\frac{1}{2+8k}\right)^{2+8k}(-1)^{2+8k}+\sin\left(\frac{\pi}{2}+k\cdot 2\pi\right)\right)\\ &= \lim_{k\to\infty}\left(\left(1+\frac{1}{2+8k}\right)^{2+8k}+1\right) = e+1.\end{aligned}$$

Hence $e+1$ is a subsequential limit of (x_n). Next let $\varepsilon > 0$ be given. Take N so that for all $n \geq N$ we have $(1+\frac{1}{n})^n < e+\varepsilon$. (This is possible since $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.) Then for all $n \geq N$ we have:

$$\left(1+\frac{1}{n}\right)^n(-1)^n + \sin\frac{n\pi}{4} < e+\varepsilon+1.$$

Hence $e+1$ has the two properties which uniquely determine $\limsup_{n\to\infty} x_n$ according to Theorem 3.17. Hence $\limsup_{n\to\infty} x_n = e+1$.

Next, we claim that $\liminf_{n\to\infty} x_n = -e-2^{-1/2}$. To prove this, note that

$$\begin{aligned}\lim_{k\to\infty} x_{5+8k} &= \lim_{k\to\infty}\left(\left(1+\frac{1}{5+8k}\right)^{5+8k}(-1)^{5+8k}+\sin\left(\frac{5\pi}{4}+k\cdot 2\pi\right)\right)\\ &= \lim_{k\to\infty}\left(-\left(1+\frac{1}{5+8k}\right)^{5+8k}-2^{-1/2}\right) = -e-2^{-1/2}.\end{aligned}$$

Hence $-e-2^{-1/2}$ is a subsequential limit of (x_n). Next let $\varepsilon > 0$ be given. Take N so that $(1+\frac{1}{n})^n < e+\varepsilon$ for all $n \geq N$ (this is possible as above). Now for every *odd* $n \geq N$ we have

$$\sin\frac{n\pi}{4} \in \left\{\sin\frac{\pi}{4}, \sin\frac{3\pi}{4}, \sin\frac{5\pi}{4}, \sin\frac{7\pi}{4}\right\} = \{-2^{-1/2}, 2^{-1/2}\},$$

and thus

$$x_n \geq -\left(1+\frac{1}{n}\right)^n - 2^{-1/2} > -e-\varepsilon-2^{-1/2}.$$

Furthermore for every *even* $n \geq N$ we have

$$x_n > 0-1 = -1.$$

Hence $x_n > -e-\varepsilon-2^{-1/2}$ holds for *all* integers $n \geq N$. Hence $-e-2^{-1/2}$ has the two properties which uniquely determine $\liminf_{n\to\infty} x_n$ according to (the lim inf analogue of) Theorem 3.17. Hence $\liminf_{n\to\infty} x_n = -e-2^{-1/2}$.

Answer: $\limsup_{n\to\infty} x_n = e+1$ and $\liminf_{n\to\infty} x_n = -e-2^{-1/2}$.

3. By integration by parts we have

$$a_n = \int_\pi^{n\pi} \frac{\sin x}{x}\,dx = \left[\frac{-\cos x}{x}\right]_\pi^{n\pi} - \int_\pi^{n\pi} \frac{\cos x}{x^2}\,dx$$
$$= -\frac{(-1)^n}{n\pi} - \frac{1}{\pi} - \int_\pi^{n\pi} \frac{\cos x}{x^2}\,dx.$$

Here the first two terms tend to 0 and $-\frac{1}{\pi}$, respectively, as $n \to \infty$; hence it now suffices to prove that the sequence

$$\left(\int_\pi^{n\pi} \frac{\cos x}{x^2}\,dx\right)_{n=1,2,\ldots} \tag{1}$$

converges in $\mathbb{R}$. We will prove this by showing that this sequence is Cauchy! Note that for any $m \geq n \geq 1$ we have

$$\left|\int_\pi^{n\pi} \frac{\cos x}{x^2}\,dx - \int_\pi^{m\pi} \frac{\cos x}{x^2}\,dx\right| = \left|\int_{n\pi}^{m\pi} \frac{\cos x}{x^2}\,dx\right|$$
$$\leq \int_{n\pi}^{m\pi} \left|\frac{\cos x}{x^2}\right|\,dx \leq \int_{n\pi}^{m\pi} \frac{1}{x^2}\,dx = \frac{1}{n\pi} - \frac{1}{m\pi} < \frac{1}{n\pi}.$$

Now for any $\varepsilon > 0$ we can take $N \in \mathbb{Z}^+$ so large that $\frac{1}{N\pi} < \varepsilon$. Then for any two integers $m \geq n \geq N$ we have, by the above computation,

$$\left|\int_\pi^{n\pi} \frac{\cos x}{x^2}\,dx - \int_\pi^{m\pi} \frac{\cos x}{x^2}\,dx\right| < \frac{1}{n\pi} \leq \frac{1}{N\pi} < \varepsilon.$$

The fact that such an N exists for every $\varepsilon > 0$ proves that the sequence in (1) is Cauchy, and we are done! □

4. Recall from my comments to the exam: $[0,\infty)$ should be corrected to $(0,\infty)$ in the statement of the problem.

Write $f_n(x) = e^{-nx}\cos n\pi x$ so that $F(x) = \sum_{n=1}^\infty f_n(x)$. We have $|f_n(x)| < e^{-nx}$ and $\sum_{n=1}^\infty e^{-nx} < \infty$ for every $x \in (0,\infty)$; hence the series defining $F(x)$ is absolutely convergent for every $x \in (0,\infty)$. We compute:

$$f_n'(x) = -e^{-nx}n\big(\cos n\pi x + \pi \sin n\pi x\big).$$

For any given $0 < a < b$, we have for all $n \geq 1$ and all $x \in [a,b]$:

$$|f_n'(x)| \leq e^{-nx}n(1+\pi) \leq e^{-na}n(1+\pi),$$

and the series $\sum_{n=1}^\infty e^{-na}n(1+\pi)$ converges since $a > 0$ (e.g. by the ratio test). Hence by the Weierstrass' M-test (Theorem 7.10), the series $\sum_{n=1}^\infty f_n'(x)$ is converges uniformly on $[a,b]$. Hence by Theorem 7.17[1], we have $F'(x) = \sum_{n=1}^\infty f_n'(x)$ for all $x \in [a,b]$. Since this is true for any given $0 < a < b$, we conclude that

$$F'(x) = \sum_{n=1}^\infty f_n'(x), \qquad \forall x \in (0,\infty); \tag{2}$$

in particular $F(x)$ is differentiable for all $x \in (0,\infty)$.

It remains to compute $F'(1)$. It seems easiest to first compute $F(x)$ for general $x > 0$. Note that $f_n(x) = \Re\big(e^{-nx(1+i\pi)}\big)$, and the series

$$G(x) = \sum_{n=1}^\infty e^{-nx(1+i\pi)}$$

is an absolutely convergent geometric series for every $x \in (0,\infty)$, since $\big|e^{-nx(1+i\pi)}\big| = e^{-nx}$. Hence, for $x \in (0,\infty)$:

$$G(x) = \frac{e^{-x(1+i\pi)}}{1-e^{-x(1+i\pi)}} = \frac{1}{1-e^{-x(1+i\pi)}} - 1,$$

and so

$$F(x) = \sum_{n=1}^\infty \Re\big(e^{-nx(1+i\pi)}\big) = \Re G(x), \qquad \forall x \in (0,\infty).$$

Therefore

$$F'(x) = \Re G'(x) = \Re\left(\frac{-(1+i\pi)e^{-x(1+i\pi)}}{(1-e^{-x(1+i\pi)})^2}\right), \qquad \forall x \in (0,\infty).$$

[1] Of course, we apply the version of Theorem 7.17 for *series*. In other words, we apply Theorem 7.17 (literally) to the sequence of functions (s_k) defined by $s_k(x) = \sum_{n=1}^k f_n(x)$.

In particular (using $e^{-i\pi} = -1$):

$$F'(1) = \Re\left(\frac{-(1+i\pi)(-e^{-1})}{(1+e^{-1})^2}\right) = \frac{e^{-1}}{(1+e^{-1})^2} = \frac{e}{(e+1)^2}.$$

Answer: $F'(1) = \frac{e}{(e+1)^2}$.

Alternative, without using any complex numbers: By (2) we have

$$F'(1) = \sum_{n=1}^{\infty} f_n'(1) = -\sum_{n=1}^{\infty} e^{-n} n(-1)^n = -\sum_{n=1}^{\infty} n(-e)^{-n}.$$

To compute this, we *repeat* the discussion which we carried out to prove the differentiability of F, but for the following function:

$$H(x) = \sum_{n=1}^{\infty} (-1)^n e^{-nx} = \sum_{n=1}^{\infty} (-e^{-x})^n = \frac{-e^{-x}}{1+e^{-x}} \qquad (x \in (0,\infty)).$$

This leads to the conclusion that H is differentiable, with

$$H'(x) = -\sum_{n=1}^{\infty} (-1)^n n e^{-nx}, \qquad \forall x \in (0,\infty).$$

In particular $H'(1) = F'(1)$. But also

$$H'(x) = \frac{e^{-x}(1+e^{-x}) + e^{-x}(-e^{-x})}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2}.$$

Hence

$$F'(1) = H'(1) = \frac{e^{-1}}{(1+e^{-1})^2} = \frac{e}{(e+1)^2}.$$

$\square$

5. NOTE: This problem lies outside the syllabus of the course, since Weierstrass' Theorem is no longer part of the syllabus (since 2019).

Assume that $f : [0,1] \to \mathbb{R}$ is continuous and that $\int_0^1 f(x)x^{2n+1}\,dx = 0$ for $n = 0, 1, 2, \ldots$. We substitute $x = \sqrt{y}$ in the integral; this gives:

$$0 = \int_0^1 f(x)x^{2n+1}\,dx = \int_0^1 f(\sqrt{y})y^{n+\frac{1}{2}}\,\frac{dy}{2\sqrt{y}} = \frac{1}{2}\int_0^1 f(\sqrt{y})y^n\,dy.$$

Hence if we define $h(y) := f(\sqrt{y})$ then we have:

$$\int_0^1 h(y)y^n\,dy = 0, \qquad \forall n \in \{0, 1, 2, \ldots\},$$

and hence

$$\int_0^1 h(y)P(y)\,dy = 0, \qquad \text{for every polynomial } P.$$

Note also that h by definition is a continuous function, $h : [0,1] \to \mathbb{R}$. But by the Weierstrass' Theorem 7.26, there exists a sequence (P_n) of polynomials such that $P_n \to h$ uniformly on $[0,1]$. Then also $hP_n \to h^2$ uniformly[2] on $[0,1]$ and hence, by Theorem 7.16,

$$\int_0^1 h(y)^2\,dy = \lim_{n\to\infty}\int_0^1 h(y)P_n(y)\,dy = \lim_{n\to\infty} 0 = 0. \tag{3}$$

Now if[3] $h(y_0) \neq 0$ for some $y_0 \in [0,1]$ then $h(y_0)^2 > 0$, and then, since h^2 is continuous, there exists some interval $[a,b] \subset [0,1]$ with $a < b$ and $y_0 \in [a,b]$, such that $h(y)^2 \geq \frac{1}{2}h(y_0)^2$ for all $y \in [a,b]$. Then (using also the fact that $h(y)^2 \geq 0$ for all y):

$$\int_0^1 h(y)^2\,dy \geq \int_a^b h(y)^2\,dy \geq \frac{1}{2}h(y_0)^2\int_a^b dy = \frac{1}{2}h(y_0)^2(b-a) > 0,$$

and this *contradicts* (3). Hence we must have $h(y_0) = 0$ for all $y_0 \in [0,1]$. This implies that $f(x) = h(x^2) = 0$ for all $x \in [0,1]$, qed.

Finally, let us prove that the corresponding statement *fails* if $[0,1]$ is replaced by $[-1,1]$. Indeed, consider for example the constant function $f(x) \equiv 1$ on $[-1,1]$. This is a continuous function and $\int_{-1}^1 f(x)x^{2n+1}\,dx = 0$ for all $n = 0, 1, 2, \ldots$. Still, f is not identically 0. $\square$

[2]Proof: Note that h is bounded, since h is a continuous function on a compact set; hence there is some $B > 0$ such that $|h(y)| \leq B$ for all $y \in [0,1]$. Now let $\varepsilon > 0$ be given. Because of $P_n \to h$ uniformly on $[0,1]$, there exists some N such that $|P_n(y) - h(y)| < \varepsilon/B$ for all $n \geq N$ and all $y \in [0,1]$. It follows that $|h(y)P_n(y) - h(y)^2| \leq |h(y)| \cdot |P_n(y) - h(y)| \leq B \cdot (\varepsilon/B) = \varepsilon$ for all $n \geq N$ and all $y \in [0,1]$. Hence hP_n indeed converges uniformly to h^2 on $[0,1]$.

[3]The discussion in the following six lines could be replaced by: "It follows from (3) and Rudin's Exc. 6:2 that $h(y) = 0$ for all $y \in [0,1]$". However I wanted to provide a detailed argument instead of referring to Rudin's Exc. 6:2.

6. Set $g(x) := \sum_{k=0}^{n} \frac{c_k}{k+1} x^{k+1}$. Note that then $g(0) = 0$ and $g'(x) = p(x)$ for all $x \in \mathbb{R}$. Also $g(1) = \sum_{k=0}^{n} \frac{c_k}{k+1} = 0$, by the assumption of the problem. Now by the Mean Value Theorem, there is some $\xi \in (0,1)$ such that $g(1) - g(0) = (1-0) \cdot g'(\xi)$, or in other words $p(\xi) = g'(\xi) = 0$. $\square$

7. Let us define the map $\phi : C([0,1]) \to C([0,1])$ by

$$(\phi(f))(x) = \int_0^1 K(x,y) f(y)\, dy \qquad (\phi \in C([0,1]),\ x \in [0,1]).$$

Let us first verify that ϕ is well-defined, i.e. that we really have $\phi(f) \in C([0,1])$ for every $f \in C([0,1])$. Thus let $f \in C([0,1])$ be given. For any and $x, x' \in [0,1]$ we have

$$\begin{aligned}
\big|\phi(f)(x) - \phi(f)(x')\big| &= \left| \int_0^1 K(x,y) f(y)\, dy - \int_0^1 K(x',y) f(y)\, dy \right| \\
&= \left| \int_0^1 \big(K(x,y) - K(x',y)\big) f(y)\, dy \right| \\
&\le \int_0^1 \big|K(x,y) - K(x',y)\big|\, |f(y)|\, dy.
\end{aligned}$$

Take $B > 0$ so that $|f(y)| \le B$ for all $y \in [0,1]$. Now K is a continuous map from the compact space $[0,1]^2$ to $\mathbb{R}$; hence by Theorem 4.19, K is uniformly continuous. Hence for any given $\varepsilon > 0$ there exists some $\delta > 0$ such that $|K(x,y) - K(x',y')| < \varepsilon/B$ for any two points $(x,y), (x',y') \in [0,1]^2$ with $\big|(x,y) - (x',y')\big| < \delta$. In particular it follows that for any $x, x' \in [0,1]$ with $|x - x'| < \delta$ we have $|K(x,y) - K(x',y)| < \varepsilon/B$ for all $y \in [0,1]$, and thus

$$\int_0^1 \big|K(x,y) - K(x',y)\big|\, |f(y)|\, dy \le \int_0^1 \frac{\varepsilon}{B} \cdot B\, dy = \varepsilon.$$

In view of the previous computation this proves that $\big|\phi(f)(x) - \phi(f)(x')\big| \le \varepsilon$ for all $x, x' \in [0,1]$ with $|x - x'| < \delta$. Hence $\phi(f)$ is (uniformly) continuous; i.e. $\phi(f) \in C([0,1])$.

Having proved that ϕ is well-defined we next prove that ϕ is a *contraction.* For any $f, g \in C([0,1])$ we have, for any $x \in [0,1]$:

$$
\begin{aligned}
|\phi(f)(x) - \phi(g)(x)| &= \left| \int_0^1 K(x,y)(f(y) - g(y))\, dy \right| \\
&\leq \int_0^1 \big|K(x,y)\big| \cdot \big|f(y) - g(y)\big|\, dy \\
&\leq \int_0^1 \frac{1}{2} \cdot d(f,g)\, dy = \frac{1}{2} d(f,g), \qquad (4)
\end{aligned}
$$

and hence
(5)

$$
d(\phi(f), \phi(g)) = \sup_{x \in [0,1]} |\phi(f)(x) - \phi(g)(x)| \leq \frac{1}{2} d(f,g) \qquad (\forall f, g \in C([0,1])).
$$

This proves that ϕ is a contraction!

Recall also that $C([0,1])$ is a *complete* metric space. (Cf. Rudin's Theorem 7.15.) Now by the Contraction Principle, it follows that ϕ has a unique fixed point in $C([0,1])$. But note that $f \in C([0,1])$ is a fixed point of ϕ if and only if f satisfies the integral equation in the problem formulation. Hence we have proved that that equation has a unique solution in $C([0,1])$! $\square$

Remark: One notes trivially that the constant function 0 is a solution of the given integral equation; hence the conclusion is that:

$$
f \in C([0,1]) \textit{ satisfies the given equation iff } f \equiv 0.
$$

In view of this it is actually overkill to refer to the Contraction Principle; instead we could just argue as follows:[4] Assume that $f \in C([0,1])$ satisfies the given equation. Then $\phi(f) = f$ and hence $d(f,0) = d(\phi(f), \phi(0)) \leq \frac{1}{2} d(f,0)$, by (5). This implies that $d(f,0) = 0$, i.e. $f = 0$ in $C([0,1])$, qed.

[4]The argument which we give here is simply the (trivial!) uniqueness part of the proof of the Contraction Principle, adapted to our special case.

8. Let $f(u,v,w) = (u+v+w-6, u^2+v^2+w^2-14)$; this is a C^1 map from $\mathbb{R}^3$ (which we view as "$\mathbb{R}^{2+1}$") to $\mathbb{R}^2$, and we have $f(1,2,3) = (0,0)$. Also the matrix of $f'(u,v,w)$ is:

$$[f'(u,v,w)] = \begin{pmatrix} (D_1f_1)(u,v,w) & (D_2f_1)(u,v,w) & (D_3f_1)(u,v,w) \\ (D_1f_2)(u,v,w) & (D_2f_2)(u,v,w) & (D_3f_2)(u,v,w) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2u & 2v & 2w \end{pmatrix}$$

The left 2×2 block of this matrix [5] equals, at $(u,v,w) = (1,2,3)$:

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix},$$

which is an invertible matrix (since its determinant is 2). Hence by the Implicit Function Theorem there exist open sets $U \subset \mathbb{R}^{2+1}$ and $W \subset \mathbb{R}$ with $(1,2,3) \in U$ and $3 \in W$, such that there exist unique functions $u : W \to \mathbb{R}$ and $v : W \to \mathbb{R}$ satisfying

$$(u(w), v(w), w) \in U \text{ and } f(u(w), v(w), w) = (0,0), \qquad \forall w \in W.$$

In other words: $(u(w), v(w))$ is a solution of the equation system in the problem, for every $w \in W$. The Implicit Function Theorem also says that these functions u and v are C^1, and that (noticing that the right 2×1 block of $[f'(1,2,3)]$ equals $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$):

$$\begin{pmatrix} u'(3) \\ v'(3) \end{pmatrix} = -\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 6 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Answer: $u'(3) = 1$ and $v'(3) = -2$.

(Alternative; outline: It is fairly easy to compute $u(w)$ and $v(w)$ explicitly: We have

$$\begin{cases} u(w) = 3 - \frac{w}{2} - \sqrt{-2 + 3w - \frac{3}{4}w^2} \\ v(w) = 3 - \frac{w}{2} + \sqrt{-2 + 3w - \frac{3}{4}w^2}, \end{cases}$$

for w in a neighborhood of 3. Hence in this neighborhood,

$$u'(w) = -\frac{1}{2} - \frac{3 - \frac{3}{2}w}{2\sqrt{-2+3w-\frac{3}{4}w^2}} \quad \text{and} \quad v'(w) = -\frac{1}{2} + \frac{3 - \frac{3}{2}w}{2\sqrt{-2+3w-\frac{3}{4}w^2}},$$

In particular this gives $u'(3) = 1$ and $v'(3) = -2$.)

[5] viz., the matrix of "$A_{\boldsymbol{x}}$" in the notation of Rudin's Theorem 9.28.