UPPSALA UNIVERSITET
Matematiska institutionen
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Tentamen i matematik
Reell Analys MA226
December 9, 2015

Tillåtna hjälpmedel: en ark papper (A4, båda sidor) med egna handskrivna anteckningar. Varje problem är värt 5 poäng. Tentamen får skrivas på svenska, eller engelska.

Allowed aids: one sheet of paper (A4, both sides) with own handwritten notes. Each problem is worth 5 points. The exam may be written in Swedish or English.

☐ **Kryssa i rutan om Du har avklarat duggan och vill tillgodoräkna dess resultat istället för att lösa två första uppgifter**. Obs! Om du skriver några lösningar till två första uppgifter och lämnar rutan blank, blir duggans resultat nollställt.

(**Check the box if you have passed the midterm and want to use its credit instead of solving two first problems.** Note that if you write any solutions to the first two problems while leaving the box blank, the midterm's result will be annulled.)

1. Consider a metric space (X, d) where X is a (abstract) set and $d(x, y) = 1$ whenever $x \neq y$ (it is called *discrete* metric space).(a) Prove that the space X is complete. (b) Prove that X is compact if and only if it consists of finitely many elements

2. Determine for what values of parameter a the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{3n}}{n^a}$$

is (a) absolutely convergent, (b) convergent, (c) divergent.

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3. Let $x_1 = 1$ and let $x_{n+1} = \frac{1}{x_n+1}$, $n \in \mathbb{N}$. Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

4. Let f be a positive Lebesgue-integrable function on $[0, 1]$. Prove that the function $F(x) = \int_{[0,x]} f(t)dt$ is continuous on $[0, 1]$. (Hint: write $F(x_n)$, $x_n \to x$, as an integral of a sequence of functions over the interval $[0, 1]$.)

5. Show that the equation

$$f(x) = \frac{1}{\pi} \int_0^1 (x - y)^7 f(y)dy + e^x$$

has a unique solution $f \in C([0, 1])$.

6. Prove that the series

$$\sum_{n=1}^{\infty} \frac{e^{-nx^2}}{n^2}$$

is convergent and defines a bounded continuous function f on $(-\infty, \infty)$. Find the $C((-\infty, \infty))$-norm of this function.

7. Prove that the following system of equations has in some neighborhood of $(x, y) = (0, 2)$ a unique solution $(u(x, y), v(x, y))$ satisfying $u(0, 2) = 1$, $v(0, 2) = 1$:

$$\begin{cases} u^2 - v^2 & = x \\ 2uv & = y \end{cases}$$

8. Let $f(x, y) = \frac{xy}{x^2+y^2}$ whenever $(x, y) \neq (0, 0)$. Find the set of limit points (=accumulation points) for f at $(0, 0)$ and determine $\limsup_{(x,y)\to(0,0)} f(x, y)$ and $\liminf_{(x,y)\to(0,0)} f(x, y)$.