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Prov i matematik
Reell analys, 1MA226
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Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. All solutions should be provided with details and appropriate justifications. No calculators are allowed.

1. A metric space (X, d) is called discrete if $d(x, y) = 1$ for all points $x \neq y$ in X. Prove that a discrete metric space (X, d) is compact if and only if the set X is finite.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:
   (a). $x_n = e^{n(-1)^n}$
   (b). $x_n = n(\sqrt[n]{n} - 1)(-1)^n + \log n + \sin \frac{2\pi n}{3}$.

3. Prove that the series

$$F(x) = \sum_{n=1}^{\infty} n^{-x} \cos n\pi x$$

converges for all $x \in (1, \infty)$, and that the function $F(x)$ is C^1 in the interval $(2, \infty)$.

4. Let the function $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \cdot 4^{-n} \leq x \leq 4^{-n} \text{ for some } n \in \{0, 1, 2, 3, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Riemann integrable on $[0, 1]$, and determine $\int_0^1 f(x)\, dx$.

5. Assume that $f : [0, 1] \to \mathbb{R}$ is a continuous function and that $\int_0^1 f(x)e^{-nx}\, dx = 0$ for $n = 0, 1, 2, \ldots$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

6. Prove that there exists an open set $U \subset \mathbb{R}^2$ with $(1, e+1) \in U$, and C^1 functions $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$, such that $u(1, e + 1) = 0$ and $v(1, e + 1) = 1$, and such that for every $(x, y) \in U$, $(u(x, y), v(x, y))$ is a solution to the following system of equations:

$$\begin{cases} u + v = x \\ e^u + e^v = y. \end{cases}$$

When this holds, determine the differentials $u'(1, e+1)$ and $v'(1, e+1)$.

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7. Prove that there exists a unique bounded sequence $(x_n) = (x_1, x_2, x_3, \ldots)$ of real numbers satisfying

$$x_n = \frac{1}{n^2} + \sum_{m=1}^{\infty} \frac{x_m}{n + 2^m}, \qquad (n = 1, 2, 3, \ldots).$$

[Hint: You may work in the metric space ℓ^∞ which consists of all bounded sequences $(x_n) = (x_1, x_2, x_3, \ldots)$ in $\mathbb{R}$, with metric

$$d((x_n), (y_n)) := \sup_{n \geq 1} |x_n - y_n|.$$

You may take it as a known fact that this metric space ℓ^∞ is complete.]

8. Let f be a continuous real function on $\mathbb{R}^2$, and let A be a linear map from $\mathbb{R}^2$ to $\mathbb{R}$. Assume that the differential $f'(x)$ exists for all $x \in \mathbb{R}^2 \setminus \{(0,0)\}$ and that

$$\lim_{x \to (0,0)} f'(x) = A$$

(convergence in the metric space $L(\mathbb{R}^2, \mathbb{R})$). Prove that then $f'(0,0)$ exists and $f'(0,0) = A$.

LYCKA TILL / GOOD LUCK!