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Matematiska Institutionen
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Prov i matematik
Reell analys, 1MA226
2019-06-15

Duration: 9.00 – 14.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. No calculators are allowed.

1. Prove that $\sup\big\{x + \frac{1}{x} \,:\, \frac{1}{2} < x < \frac{3}{2}\big\} = \frac{5}{2}$.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:
   (a). $x_n = (1 + (-1)^n)n$.
   (b). $x_n = (-1)^n(1 + \frac{1}{n})^n + \sin\frac{2\pi n}{3}$.

3. Prove that the series
$$F(x) = \sum_{n=1}^{\infty} e^{-nx}(\log n - \sin nx)$$
converges for all $x > 0$, and that the function $F : (0, \infty) \to \mathbb{R}$ is C^1.

4. Give an example of an open cover of the interval $(0, 1]$ which has no finite subcover. (Note: You must prove that your open cover indeed does not have any finite subcover.)

5. Show that the integral equation
$$f(x) = \frac{1}{2}\int_x^1 (y - x)f(y)\,dy + xe^{x^2}$$
has a unique solution $f \in C([0, 1])$.

6. Prove that there exists an open set $U \subset \mathbb{R}^2$ with $(2, e) \in U$, and C^1 functions $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$, such that $u(2, e) = 0$ and $v(2, e) = 1$, and such that for every $(x, y) \in U$, $(u(x, y), v(x, y))$ is a solution to the following system of equations:
$$\begin{cases} e^u + v = x \\ u + e^v = y. \end{cases}$$
When this holds, determine the differentials $u'(2, e)$ and $v'(2, e)$.

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7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

(a). Compute $D_1 f(0,0)$ and $D_2 f(0,0)$

(b). Prove that f is not differentiable at $(0,0)$.

8. Set $A = \{1^{-1}, 2^{-1}, 3^{-1}, \ldots\}$, and let $f : \mathbb{R} \to \mathbb{R}$ be the indicator function of A, i.e.

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that f is Riemann integrable on $[0,1]$, and determine $\int_0^1 f(x)\,dx$.

LYCKA TILL / GOOD LUCK!