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Prov i matematik
Reell analys, 1MA226
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Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. Permitted aids: one sheet of paper (A4, both sides) with own handwritten notes. No calculators are allowed.

1. Let X be a metric space and let A be a subset of X. Assume that A is not closed. Prove that there exists a Cauchy sequence (x_n) in A which does not converge to any point in A.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:

(a). $x_n = 1 + (-1)^n + 2(-1)^{[n/2]}$.

(Here $[\alpha]$ denotes the integer part of α, i.e. the largest integer $\leq \alpha$.)

(b). $x_n = n^3 \sin\left(\frac{1}{n}\right) + (-1)^n n^2$.

3. Prove that the series $F(x) = \sum_{n=1}^{\infty} e^{-nx} \sin(n^3 x)$ converges for all $x > 0$, and that the function $F : (0, \infty) \to \mathbb{R}$ is C^1.

4. Let $f : [0, 2] \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 2 & \text{if } x \in \mathbb{Q} \cap (1, 2] \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Determine (with proof) the upper and lower Riemann integrals $\overline{\int_0^2} f(x)\,dx$ and $\underline{\int_0^2} f(x)\,dx$. Is f Riemann integrable?

5. Let $X = C([0, 1])$, equipped with its standard metric, i.e.

$$d(f, g) = \sup\{|f(x) - g(x)| \,:\, x \in [0, 1]\} \qquad \text{for all } f, g \in X.$$

Give an example of a bounded sequence in X which does not have any convergent subsequence. (Note: You must prove that your sequence is indeed bounded and does not have any convergent subsequence.)

6. Prove that there exists an open set $U \subset \mathbb{R}^2$ with $(4, 0) \in U$, and C^1 functions $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$, such that $u(4, 0) = 2$ and $v(4, 0) = 0$, and such that for every $(x, y) \in U$, $(u(x, y), v(x, y))$ is a solution to the following system of equations:

$$\begin{cases} u^2 e^v = x \\ v e^u = y. \end{cases}$$

When this holds, determine the differentials $u'(4, 0)$ and $v'(4, 0)$.

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7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{x^3y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

(a). Prove that f is continuous at $(0,0)$.
(b). Prove that f is not differentiable at $(0,0)$.

8. Let F be a family of functions from $[0,1]$ to $\mathbb{R}$. Assume that

$$\text{(I)} \qquad \forall x \in [0,1] : \forall \varepsilon > 0 : \exists \delta > 0 : \forall f \in F : \forall y \in [0,1] : \\ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Prove that

$$\text{(II)} \qquad \forall \varepsilon > 0 : \exists \delta > 0 : \forall f \in F : \forall x, y \in [0,1] : \\ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

LYCKA TILL / GOOD LUCK!