Exam, Real analysis, 1MA226, 2020-03-16.
Solution suggestions.

1. Since A is not closed, there exists a point $x \in X \setminus A$ which is a limit point of A. The fact that x is a limit point of A means that for every $r > 0$ we have $N_r(x) \cap A \neq \emptyset$. [1] In particular, for every $n \in \mathbb{Z}^+$ we can choose a point x_n in $N_{1/n}(x) \cap A$. The sequence (x_n) constructed in this way lies in A (viz., $x_n \in A$ for all n), and satisfies

$$\lim_{n\to\infty} x_n = x.$$

(Proof: Given $\varepsilon > 0$, take $N \in \mathbb{Z}^+$ so large that $1/N < \varepsilon$. Then for every $n \geq N$ we have $x_n \in N_{1/n}(x)$, and so $d(x_n, x) < 1/n \leq 1/N < \varepsilon$.)

Now $\lim_{n\to\infty} x_n = x$ implies that (x_n) is Cauchy (Theorem 3.11(a)). Also, since $\lim_{n\to\infty} x_n = x$ and $x \notin A$ (and using Theorem 3.2(b)), (x_n) does not converge to any point in A. □

[1] More precisely from the definition of limit point, we have $N_r(x) \cap A \setminus \{x\} \neq \emptyset$; but this is equivalent with $N_r(x) \cap A \neq \emptyset$ since $x \notin A$.

2. (a). (I) For $n = 1, 5, 9, \dots$ (i.e., $n = 4k + 1$ with $k = 0, 1, 2, \dots$), n is odd while $[n/2]$ is even, and so $x_n = 1 + (-1) + 2 = 2$.

(II) For $n = 2, 6, 10, \dots$ (i.e., $n = 4k + 2$ with $k = 0, 1, 2, \dots$), n is even while $[n/2]$ is odd, and so $x_n = 1 + 1 - 2 = 0$.

(III) For $n = 3, 7, 11, \dots$ (i.e., $n = 4k + 3$ with $k = 0, 1, 2, \dots$), both n and $[n/2]$ are odd, and so $x_n = 1 + (-1) - 2 = -2$

(IV) For $n = 4, 8, 12, \dots$ (i.e., $n = 4k + 4$ with $k = 0, 1, 2, \dots$), both n and $[n/2]$ are even, and so $x_n = 1 + 1 + 2 = 4$.

It follows from these observations that

$$\limsup_{n\to\infty} x_n = \max(2, 0, -2, 4) = 4$$

and

$$\liminf_{n\to\infty} x_n = \min(2, 0, -2, 4) = -2.$$

(b). For all $n \geq 1$ we have $0 < \frac{1}{n} \leq 1$ and hence (since the function $\sin x$ is increasing for $0 \leq x \leq \pi/2$),

$$0 < \sin\left(\frac{1}{n}\right) \leq \sin 1 < 1.$$

It follows that for n *even* we have

$$x_n > (-1)^n n^2 = n^2 \to +\infty \qquad \text{as } n \to \infty.$$

This implies that

$$\limsup_{n\to\infty} x_n = +\infty.$$

Next, for n *odd* we have, using the Taylor expansion of the function $\sin x$ near $x = 0$:

$$\begin{aligned} x_n &= n^3 \sin\left(\frac{1}{n}\right) - n^2 \\ &= n^3\Big(n^{-1} - \frac{1}{6}n^{-3} + O(n^{-5})\Big) - n^2 \\ &= n^2 - \frac{1}{6} + O(n^{-2}) - n^2 \\ &= -\frac{1}{6} + O(n^{-2}) \to -\frac{1}{6} \qquad \text{as } n \to \infty. \end{aligned}$$

Since $x_n \to +\infty$ as $n \to \infty$ through even integers and $x_n \to -\frac{1}{6}$ as $n \to \infty$ through odd integers, we conclude that

$$\liminf_{n\to\infty} x_n = -\frac{1}{6}.$$

3. Let $[a,b]$ be any real interval with $0 < a < b$. Note that

$$\left| e^{-nx} \sin(n^3 x) \right| \le e^{-nx} \le e^{-na}, \qquad \forall n \in \mathbb{Z}^+,\ x \in [a,b].$$

Furthermore the series $\sum_{n=1}^{\infty} e^{-na}$ converges. Hence by Weierstrass' M-test, we conclude that the series defining $F(x)$ is uniformly convergent on $[a,b]$.

Hence it follows that F is well-defined and continuous in the interval $[a,b]$. Since this is true for any $0 < a < b$, it follows that F is well-defined and continuous in the whole interval $(0,\infty)$.

Next consider the series obtained by formally differentiating the series for $F(x)$ term by term, i.e.:

$$\sum_{n=1}^{\infty} e^{-nx}\big(-n \sin(n^3 x) + n^3 \cos(n^3 x)\big). \tag{1}$$

We claim that this series is uniformly convergent on any interval $[a,b]$ with $0 < a < b$. Indeed, for all $n \in \mathbb{Z}^+$ and $x \in [a,b]$ we have

$$\left| e^{-nx}\big(-n \sin(n^3 x) + n^3 \cos(n^3 x)\big) \right| \le e^{-na}\,(n + n^3).$$

Furthermore, for any $a > 0$ we have

$$\lim_{n\to\infty} \frac{e^{-(n+1)a}\,\big((n+1) + (n+1)^3\big)}{e^{-na}\,(n+n^3)} = e^{-a} < 1,$$

and hence by the ratio test, the series $\sum_{n=1}^{\infty} e^{-na}\,(n+n^3)$ converges. Using the above facts in combination with Weierstrass' M-test, we conclude that the series in (1) is indeed uniformly convergent on $[a,b]$. Hence by Rudin's Thm. 7.17, we have that $F'(x)$ exists for all $x \in [a,b]$, and

$$F'(x) = \sum_{n=1}^{\infty} e^{-nx}\big(-n \sin(n^3 x) + n^3 \cos(n^3 x)\big).$$

The uniform convergence pointed out above (together with the fact that each term is a continuous function of x) implies that this function is continuous in $[a,b]$. Hence F is C^1 in $[a,b]$. Since this is true for any $0 < a < b$, we conclude that F is C^1 in the whole interval $(0,\infty)$. $\square$

4. Consider an arbitrary partition $P = (x_i)_{i=0}^n$ of $[0,2]$ (thus $0 = x_0 < x_1 < \cdots < x_n = 2$). For each $i \in \{1,\ldots,n\}$, using $x_{i-1} < x_i$ and the fact that both the sets $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$, it follows that

$$M_i := \sup_{x\in[x_{i-1},x_i]} f(x) = \begin{cases} 2 & \text{if } x_i > 1 \\ 1 & \text{if } x_i \le 1. \end{cases}$$

and

$$m_i := \inf_{x\in[x_{i-1},x_i]} f(x) = 0.$$

Hence, letting j be the largest index in $\{0,1,\ldots,n-1\}$ for which $x_j \le 1$, we have:

$$U(P,f) = \sum_{i=1}^n M_i(x_i - x_{i-1}) = 1\cdot x_j + 2\cdot(2-x_j) = 4 - x_j.$$

Hence $U(P,f) \ge 4-1 = 3$ for *all* partitions P of $[0,2]$. On the other hand, of course there exist partitions P of $[0,2]$ for which $x_j = 1$, giving $U(P,f) = 3$. Hence

$$\overline{\int_0^2} f(x)\,dx = \inf\bigl\{U(P,f)\ :\ P \text{ is a partition of } [0,2]\bigr\} = 3.$$

On the other hand, using the fact that $m_i = 0$ for all i and all partitions P, we obtain

$$L(P,f) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = 0$$

for *all* partitions P. Hence

$$\underline{\int_0^2} f(x)\,dx = \sup\bigl\{L(P,f)\ :\ P \text{ is a partition of } [0,2]\bigr\} = 0.$$

The fact that $\overline{\int_0^2} f(x)\,dx \ne \underline{\int_0^2} f(x)\,dx$ implies that f is *not* Riemann integrable. $\square$

5. For example, consider the sequence (f_n) in X, where

$$f_n(x) = \max(0, 1 - nx), \qquad (n = 1, 2, 3, \ldots).$$

Note that $f_n(x) \in [0,1]$ for all $n \in \mathbb{Z}^+$ and all $x \in [0,1]$. Hence $d(f_n, 0) \leq 1$ for all $n \in \mathbb{Z}^+$ (where "0" stands for the constant function 0), and this shows that our sequence (f_n) is *bounded*.

Now *assume* that we have a convergent subsequence (f_{n_m}), where $1 \leq n_1 < n_2 < \cdots$. This means that there exists some $f \in X$ such that $\lim_{j\to\infty} f_{n_j} = f$. But convergence in the space $X = C([0,1])$ is the same as uniform convergence, i.e. we have $f_{n_j} \to f$ uniformly on $[0,1]$ as $j \to \infty$. This in turn implies that $f_{n_j} \to f$ *pointwise* on $[0,1]$ as $j \to \infty$, i.e. that for every $x \in [0,1]$ we have

$$f(x) = \lim_{j\to\infty} f_{n_j}(x) = \lim_{j\to\infty} \max(0, 1 - n_j x).$$

However if $x > 0$ then the last limit above is seen to be 0, since $1 - n_j x \to -\infty$ as $j \to \infty$. On the other hand if $x = 0$ then the limit is seen to be 1, since $1 - n_j \cdot 0 = 1$ for all j. Hence:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0,1]. \end{cases}$$

However this function f is not continuous, i.e. $f \notin X$. This is a contradiction! Hence the sequence (f_n) does not have any convergent subsequence. $\square$

(Another example is given in Rudin's Ex. 7.21.)

6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$F(u,v) = (u^2 e^v, ve^u).$$

Note that F is C^1. We compute:

$$[F'(u,v)] = \begin{bmatrix} 2ue^v & u^2 e^v \\ ve^u & e^u \end{bmatrix}.$$

In particular

$$[F'(2,0)] = \begin{bmatrix} 4 & 4 \\ 0 & e^2 \end{bmatrix},$$

which is non-singular. Hence by the *Inverse Function Theorem* there exists an open set $V \subset \mathbb{R}^2$ which contains the point $(2,0)$, such that $F|_V$ is C^1, $U := F(V)$ is open, and $G := (F|_V)^{-1} : U \to V$ is C^1.

By the definition of $G = (F|_V)^{-1}$ we have $F(G(x,y)) = (x,y)$ for all $(x,y) \in U$. In other words:

$$\begin{cases} G_1(x,y)^2 e^{G_2(x,y)} = x \\ G_2(x,y) e^{G_1(x,y)} = y. \end{cases} \qquad \forall (x,y) \in U.$$

Also $G(F(2,0)) = (2,0)$, i.e. $G(4,0) = (2,0)$. This means that if we write $u = G_1 : U \to \mathbb{R}$ and $v = G_2 : U \to \mathbb{R}$ then the functions u and v have all the properties required in the problem formulation!

By the chain rule we also have $F'(G(x,y)) \cdot G'(x,y) = I$ for all $(x,y) \in U$; thus in particular $F'(2,0) \cdot G'(4,0) = I$, or in other words:

$$[G'(4,0)] = [F'(2,0)]^{-1} = \begin{bmatrix} 4 & 4 \\ 0 & e^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & -e^{-2} \\ 0 & e^{-2} \end{bmatrix}.$$

But we also know

$$[G'] = \begin{bmatrix} D_1 G_1 & D_2 G_1 \\ D_1 G_2 & D_2 G_2 \end{bmatrix} = \begin{bmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{bmatrix}.$$

Hence:

$$[u'(4,0)] = \left[\frac{1}{4}, -e^{-2}\right] \quad \text{and} \quad [v'(4,0)] = \left[0, e^{-2}\right].$$

$\square$

7.

(a). For any $(x,y) \neq (0,0)$ we have

$$\left|\frac{x^3y^2}{(x^2+y^2)^2}\right| \leq \frac{|x|(x^2+y^2)^2}{(x^2+y^2)^2} = |x|.$$

Hence f is continuous at $(0,0)$. [Details: Given $\varepsilon > 0$ we have for all (x,y) in $N_{(0,0)}(\varepsilon)$: $|f(x,y) - f(0,0)| \leq |x| < \varepsilon$. Done!]

(b). Assume that f is differentiable at $(0,0)$. Then by Theorem 9.17, the partial derivatives $a = D_1 f(0,0)$ and $b = D_2 f(0,0)$ exist, and we have $[f'(0,0)] = [a \quad b]$. But note that $f(x,0) = 0$ for all $x \in \mathbb{R}$ and $f(0,y) = 0$ for all $y \in \mathbb{R}$; hence we have $a = b = 0$, and so

$$[f'(0,0)] = [0 \quad 0].$$

Hence the assumption that f is differentiable at $(0,0)$ means that

$$\lim_{(x,y)\to(0,0)} \frac{\left|f(x,y) - f(0,0) - 0x - 0y\right|}{\sqrt{x^2+y^2}} = 0,$$

or equivalently:

$$\lim_{(x,y)\to(0,0)} \frac{\left|x^3y^2\right|}{(x^2+y^2)^{5/2}} = 0.$$

In particular this implies, by letting $(x,y) \to (0,0)$ along the line $y = x$:

$$\lim_{x\to 0} \frac{|x^5|}{(2x^2)^{5/2}} = 0. \tag{2}$$

However for all $x > 0$ we have $\frac{|x^5|}{(2x^2)^{5/2}} = \frac{x^5}{2^{5/2}x^5} = 2^{-5/2}$; hence

$$\lim_{x\to 0^+} \frac{|x^5|}{(2x^2)^{5/2}} = 2^{-5/2}. \tag{3}$$

Together, (2) and (3) give a *contradiction!*

This proves that f is *not* differentiable at $(0,0)$. $\square$

8. (The following proof is closely inspired by Rudin's proof of Theorem 4.19.)

Let $\varepsilon > 0$ be given. Then by (I) (applied with $\varepsilon/2$ in place of ε), for each $x \in [0,1]$ there exists a number $\delta_x > 0$ such that

$$(*) \qquad \forall x \in [0,1]: \quad \forall f \in F: \quad \forall y \in N_{\delta_x}(x) \cap [0,1]: \quad |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

For each $x \in [0,1]$ we set

$$I_x := N_{\delta_x/2}(x) \cap [0,1].$$

Then for every $x \in [0,1]$ we have $x \in I_x$ and I_x is an open subset of $[0,1]$. Hence the family $\{I_x \ : \ x \in [0,1]\}$ is an open cover of $[0,1]$. Hence, since $[0,1]$ is compact, there exists a finite subcover; in other words, there exists a finite set of points $x_1, \ldots, x_n \in [0,1]$ such that

$$(**) \qquad [0,1] = I_{x_1} \cup \cdots \cup I_{x_n}.$$

Now set

$$\delta := \frac{1}{2} \min(\delta_{x_1}, \ldots, \delta_{x_n}) > 0.$$

With this δ, we claim that

$$(***) \qquad \forall f \in F: \quad \forall x, y \in [0,1]: \qquad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

To prove this, consider any $f \in F$ and any $x, y \in [0,1]$, and assume that $|x - y| < \delta$. By (**), there is some $m \in \{1, \ldots, n\}$ such that $x \in I_{x_m}$, i.e. $|x - x_m| < \delta_{x_m}/2$. Then also

$$|y - x_m| \le |y - x| + |x - x_m| < \delta + \frac{\delta_{x_m}}{2} \le \frac{\delta_{x_m}}{2} + \frac{\delta_{x_m}}{2} = \delta_{x_m},$$

Hence both x and y lie in $N_{\delta_{x_m}}(x_m)$, and hence by (*) (applied with x_m in place of x),

$$|f(x) - f(y)| \le |f(x) - f(x_m)| + |f(y) - f(x_m)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence we have proved (***).

We have proved that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that (***) holds. This means that (II) holds! $\square$

(See next page for an alternative solution.)

Alternative solution.

Assume that (I) holds but (II) does *not* hold, that is, there exists some $\varepsilon > 0$ such that

$$\forall \delta > 0 : \exists f \in F : \exists x, y \in [0,1] : |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon.$$

We will prove that this leads to a contradiction.

Applying the last statement with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, it follows that there exist a sequence (f_n) in F and sequences (x_n) and (y_n) in $[0,1]$, such that

$$(*) \qquad \forall n \in \mathbb{Z}^+ : \qquad |x_n - y_n| < \frac{1}{n} \text{ and } |f_n(x_n) - f_n(y_n)| \geq \varepsilon.$$

By Theorem 3.6, since $[0,1]$ is compact, the sequence (x_n) has a convergent subsequence i.e. we can find $1 \leq n_1 < n_2 < \cdots$ such that the limit

$$x = \lim_{k \to \infty} x_{n_k}$$

exists in $[0,1]$. Applying now (I) to *this* point x, and with $\varepsilon/2$ in place of ε, we conclude that there exists some $\delta > 0$ such that

$$(**) \qquad \forall f \in F : \forall y \in [0,1] : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \tfrac{1}{2}\varepsilon.$$

Now fix k so large that

$$\frac{1}{n_k} < \tfrac{1}{2}\delta \quad \text{and} \quad |x_{n_k} - x| < \tfrac{1}{2}\delta$$

(this is possible since $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} x_{n_k} = x$). For this k, we have

$$|x_{n_k} - y_{n_k}| < \frac{1}{n_k} < \tfrac{1}{2}\delta$$

(by (*)), and combining this with $|x_{n_k} - x| < \frac{1}{2}\delta$, we conclude that $|y_{n_k} - x| < \delta$. Hence by (**) we have both $|f_{n_k}(x_{n_k}) - f_{n_k}(x)| < \frac{1}{2}\varepsilon$ and $|f_{n_k}(y_{n_k}) - f_{n_k}(x)| < \frac{1}{2}\varepsilon$. Hence

$$|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| < \varepsilon.$$

This is a *contradiction* against (*) above. $\square$