UPPSALA UNIVERSITET
Matematiska institutionen
Douglas Lundholm

Hemtentamen i matematik
Reell analys 1MA226
15 mars 2021

Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. Permitted aids: Course material (including books), lecture notes, old problems and solutions.

1. Give examples or claim non-existence (with brief motivations) of:

a) A bounded subset of $\mathbb{R}^2$ with the same cardinality as $\mathbb{R}$.
b) A bounded metric space which is complete but not compact.
c) A bounded real-valued function on $[0,1]$ which is not a derivative.
d) A bounded subset of $C([-1,1]\times[-1,1])$ which is not equicontinuous.
e) A discontinuous linear map.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:

a) $x_n = \tan((1+2n)\pi/4)$
b) $x_n = \sum_{k=0}^{n} 4^k (k!)^{(-1)^n}$

3. Prove that the series $F(x) = \sum_{n=1}^{\infty} \frac{n^2+x^4}{n^4+x^2}$ converges for all $x \in \mathbb{R}$, that the function $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and give an expression for $F'(x)$.

4. On the vector space ℓ^∞ of bounded real sequences $x = (x_n)_{n=0}^\infty$ we have defined the addition $\ell^\infty \times \ell^\infty \to \ell^\infty$,

$$x + y := (x_n + y_n)_{n=0}^\infty,$$

with unit $\mathbf{0} = (0)_{n=0}^\infty$, scalar multiplication $\mathbb{R} \times \ell^\infty \to \ell^\infty$,

$$\alpha x := (\alpha x_n)_{n=0}^\infty,$$

and we can also define the pointwise multiplication $\ell^\infty \times \ell^\infty \to \ell^\infty$,

$$xy := (x_n y_n)_{n=0}^\infty,$$

with unit $\mathbf{1} = (1)_{n=0}^\infty$. On the subset $\mathcal{E} = \{x \in \ell^\infty : 2^{-1} < x_n < 2 \; \forall n \in \mathbb{N}\}$ define a map

$$f : \mathcal{E} \to \mathcal{E}, \qquad f(x) := \exp(-x^{-1}) = (e^{-x_n^{-1}})_{n=0}^\infty.$$

Show that f is differentiable on $\mathcal{E}$ and compute its derivative at $x = \left(2^{\frac{1}{2+n}}\right)_{n=0}^\infty$.
(Solving the problem with ℓ^∞ replaced by $\mathbb{R}$ is worth 1p.)

– Also see next page! / Var god vänd! –

5. Show that there exists a unique function $f \in C([-1,1])$ that solves the equation

$$4f(x) = \pi + f(\sin(\pi x)) + \int_{-\infty}^{\infty} e^{-|x-y|} f\left(\frac{2}{\pi}\arctan y\right) dy, \qquad x \in [-1,1].$$

6. Consider for $a, x, y, u, v \in \mathbb{R}$ the system of equations

$$\begin{cases} xye^{u+v} &= 1 \\ x+y+u+av &= 2 \end{cases}$$

a) Determine for which values of a that u and v can be solved as continuously differentiable functions of x and y locally around $(x,y) = (1,1)$.

b) Compute for any choice of a and (u,v) found in a) a linear approximation to the transformation

$$(x,y) \mapsto \mathbf{f}(x,y) = (u(x,y), v(x,y))$$

at the point $(x,y) = (1,1)$.

7. Determine if the set $\mathscr{R}$ of Riemann integrable functions on the interval $[a,b]$, $a < b$, is a closed subset of the space of bounded real-valued functions $\ell^\infty([a,b];\mathbb{R})$

a) In the uniform topology? (where $f_n \to f$ iff $\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0$)

b) In the pointwise topology? (where $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in [a,b]$)

8. Let $(f_n)_{n=1}^\infty$ be a sequence of continuously differentiable functions on $\mathbb{R}$ such that the sequence of derivatives (f_n') is uniformly bounded and $f_n(0) = 0$ for all n. Show that there exists a subsequence of (f_n) such that the sequence of functions (F_n) defined by

$$F_n(x) := \frac{1}{n} \sum_{\substack{k\in\mathbb{Z} \\ 0\le k\le xn}} f_n(k/n), \qquad x \in \mathbb{R},\ n = 1,2,3,\dots$$

converges uniformly for $x \in [0,1]$.

Good luck! / Lycka till!