

SOLUTION SUGGESTIONS.

1. a) The set $A = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = 0\} \subseteq \mathbb{R}^2$ is bounded and in bijective correspondence to the interval $I = (0, 1) \subseteq \mathbb{R}$. Hence $\text{card } A = \text{card } I = \text{card } \mathbb{R}$ by Exercise 2.4b.
- b) Given any infinite set X , the corresponding discrete metric space (X, d) has the required properties by Exercise 4.20. Alternative: the closed unit ball in ℓ^p , by Theorem 4.70.
- c) By Darboux' theorem, see Section 5.3.4, any function which does not take intermediate values is not a derivative, such as $f(x) = 0$ for $0 \leq x < 1/2$ and $f(x) = 1$ for $1/2 \leq x \leq 1$.
- d) Take for example $F_n(x, y) = f_n(x)f_n(y)$ where (f_n) in $C([-1, 1])$ is bounded but not equicontinuous, such as $f_n(x) = \max\{1 - n|x|, 0\}$. Note that for $n > 1/\delta$

$$F_n(0, 0) - F_n(\delta, \delta) = 1 - f_n(\delta)^2 = 1.$$

- e) Example 5.23 or Exercise 5.6 provide such examples. (Infinite dimensions are necessary by Exercise 5.5.)
2. a) For n even we have $x_n = \tan((1 + 2n)\pi/4) = \tan(\pi/4) = 1$ and for n odd $x_n = \tan(-\pi/4) = -1$. Therefore, any subsequence $(x_{n'})$ which eventually takes only even indices $n' \in 2\mathbb{N}$ (or odd, $2\mathbb{N} + 1$) will have the limit 1 (or -1), while any other subsequence is such that there are both infinitely many even and odd indices, and cannot have a limit. Therefore $\liminf_{n \rightarrow \infty} x_n = -1$ and $\limsup_{n \rightarrow \infty} x_n = +1$.
 - b) If n is even then

$$x_n = \sum_{k=0}^n 4^k (k!)^{(-1)^n} = \sum_{k=0}^n 4^k (k!) \geq 4^n n!$$

which diverges to $+\infty$, while if n is odd,

$$x_n = \sum_{k=0}^n \frac{4^k}{k!} \rightarrow e^4, \text{ as } n \rightarrow \infty.$$

Again, any subsequence $(x_{n'})$ which eventually takes only odd indices for n' will have the limit e^4 , while if infinitely many even indices are taken then also larger and larger values are taken. For subsequences that eventually only take even indices the values diverge. Note also that $x_{n'} \geq \sum_{k=0}^{n'} 4^k (k!)^{-1} \rightarrow e^4$ for any subsequence $n' \in \mathbb{N}$. Therefore $\liminf_{n \rightarrow \infty} x_n = e^4$ and $\limsup_{n \rightarrow \infty} x_n = +\infty$.

3. We write for convenience $F(x) = \sum_{n=1}^{\infty} f_n(x)$ where

$$f_n(x) := \frac{n^2 + x^4}{n^4 + x^2}, \quad n \geq 1, x \in \mathbb{R}.$$

Note that $f_n \in C(\mathbb{R})$ but is not bounded ($\|f_n\|_{\infty} = \infty$). However if we consider an interval $[-R, R]$ with fixed $R \in \mathbb{R}^+$, then $f_n \in C([-R, R])$ with

$$|f_n(x)| \leq \frac{n^2 + x^4}{n^4} \leq \frac{1}{n^2} + \frac{R^4}{n^4},$$

for any $x \in [-R, R]$, so (with norms on $C([-R, R]) = C_b([-R, R])$)

$$\|f_n\| \leq \frac{1}{n^2} + \frac{R^4}{n^4},$$

and since these are summable we get a convergent uniform bound

$$\|F\| \leq \sum_{n=1}^{\infty} \|f_n\| \leq K_R$$

for some constant $K_R < \infty$.

By completeness of the Banach space $C([-R, R])$ (Theorem 6.8) and the majorization theorem for series (Theorem 4.66), $F = \sum_{n=1}^{\infty} f_n$ converges uniformly on $[-R, R]$ to its pointwise limit (namely $F_N := \sum_{n=1}^N f_n$ is uniformly Cauchy).

Now consider differentiating the finite sum F_N , with each term

$$f'_n(x) = \frac{4x^3}{n^4 + x^2} - 2x \frac{n^2 + x^4}{(n^4 + x^2)^2} = \frac{4x^3 n^4 + 2x^5 - 2xn^2}{(n^4 + x^2)^2},$$

$$|f'_n(x)| \leq \frac{4R^3 n^4 + 2R^5 + 2Rn^2}{n^8} \quad \text{for } x \in [-R, R].$$

Therefore

$$\|f'_n\| \leq 4R^3 n^{-4} + 2R^5 n^{-8} + 2Rn^{-6},$$

and the function $F'_N = \sum_{n=1}^N f'_n \in C([-R, R])$ has a convergent uniform bound

$$\|F'_N\| \leq \sum_{n=1}^{\infty} \|f'_n\| \leq K'_R < \infty.$$

Again, by completeness and majorization, F'_N converges uniformly on $[-R, R]$, and by Theorems 6.8/9 and 6.14 we conclude that uniformly $F_N \rightarrow F \in C^1([-R, R])$ and

$$F' = \lim_{N \rightarrow \infty} F'_N = \sum_{n=1}^{\infty} f'_n.$$

Furthermore, since R was arbitrary we have this formula pointwise on \mathbb{R} , and $F \in C^1(\mathbb{R})$.

4. We note that with the given definitions, $(\ell^\infty, +, \cdot, \|\cdot\|_\infty)$ becomes a commutative Banach ring. We also note that

$$\mathcal{E} = \{x \in \ell^\infty : x_n \in (1/2, 2) \forall n \in \mathbb{N}\} \subseteq \Omega := \{x \in \ell^\infty : x_n \neq 0 \forall n, \|x^{-1}\|_\infty < \infty\},$$

the latter set being the invertible elements of ℓ^∞ .

By Theorem 5.15 we have that the inverse map $g: \Omega \rightarrow \Omega$, $g(x) := x^{-1}$, is differentiable on Ω ,

$$g(x+h) = g(x) + g'(x)h + o(h),$$

with derivative $g'(x) \in \text{Hom}(\ell^\infty, \ell^\infty)$,

$$g'(x)h = -x^{-1}hx^{-1} = -x^{-2}h, \quad x \in \Omega, h \in \ell^\infty,$$

$$\|g'(x)\|_{\text{op}} = \sup_{\|h\|_\infty=1} \|g'(x)h\|_\infty = \|x^{-2}\|_\infty = \|x^{-1}\|_\infty^2 < \infty.$$

Furthermore (see Theorem 4.76), $\exp: \ell^\infty \rightarrow \ell^\infty$, $\exp(x) = (e^{x_n})_{n=0}^\infty$, is everywhere differentiable,

$$\exp(x+h) = \exp(x)\exp(h) = \exp(x)(1+h+O(h^2)) = \exp(x) + \exp'(x)h + O(h^2),$$

with derivative $\exp'(x) = \exp(x)1 \in \text{Hom}(\ell^\infty, \ell^\infty)$, $\|\exp'(x)\|_{\text{op}} = \|\exp(x)\|_\infty = e^{\|x\|_\infty}$.

Thus $f(x) = \exp(-g(x))$ is well defined on \mathcal{E} and by the chain rule also differentiable,

$$\begin{aligned} f(x+h) &= \exp(-g(x+h)) = \exp(-g(x) + x^{-2}h + o(h)) = \exp(-g(x))\exp(x^{-2}h + o(h)) \\ &= f(x)(1 + x^{-2}h + o(h) + O(h^2)) = f(x) + f'(x)h + o(h), \end{aligned}$$

with derivative

$$f'(x) = \exp(-x^{-1})x^{-2}.$$

For $x_n = 2^{\frac{1}{2+n}}$ we have $x_n^{-1} = 2^{-\frac{1}{2+n}}$ and

$$f'(x) = \left(e^{-x_n^{-1}} x_n^{-2} \right)_{n=0}^\infty = \left(e^{-2^{-\frac{1}{2+n}}} 2^{-\frac{2}{2+n}} \right)_{n=0}^\infty,$$

which is a bounded operator on ℓ^∞ :

$$\|f'(x)\|_{\text{op}} = \sup_{n \in \mathbb{N}} \left\{ e^{-2^{-\frac{1}{2+n}}} 2^{-\frac{2}{2+n}} \right\} = e^{-1}.$$

5. Use Banach's Fixpoint Theorem (BFT) with the function $\Phi: C([-1, 1]) \rightarrow C([-1, 1])$,

$$[\Phi(f)](x) := \frac{\pi}{4} + \frac{1}{4}f(\sin(\pi x)) + \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-y|} f\left(\frac{2}{\pi} \arctan y\right) dy, \quad x \in [-1, 1].$$

Note that the arguments of f are indeed in the interval $[-1, 1]$, and we have compositions with uniformly continuous, bounded functions. The generalized integral

$$F(x) := \int_{-\infty}^{\infty} e^{-|x-y|} f\left(\frac{2}{\pi} \arctan y\right) dy = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-|x-y|} f\left(\frac{2}{\pi} \arctan y\right) dy$$

is absolutely convergent, and furthermore is evidently continuous in x if we write

$$\begin{aligned} F(x) &= \int_{-\infty}^x e^{-(x-y)} f\left(\frac{2}{\pi} \arctan y\right) dy + \int_x^{\infty} e^{-(y-x)} f\left(\frac{2}{\pi} \arctan y\right) dy \\ &= e^{-x} \int_{-\infty}^x e^y f\left(\frac{2}{\pi} \arctan y\right) dy + e^x \int_x^{\infty} e^{-y} f\left(\frac{2}{\pi} \arctan y\right) dy. \end{aligned}$$

Now, since the space $C([-1, 1])$ is complete by Theorem 6.8, if we verify that Φ is a strict contraction then by Theorem 7.1 (BFT) there exists a unique $f \in C([-1, 1])$ which solves $f = \Phi(f)$.

Take $f, g \in C([-1, 1])$ and consider $h := f - g$, then by the triangle inequality

$$\begin{aligned} |\Phi(f)(x) - \Phi(g)(x)| &= \left| \frac{1}{4}h(\sin(\pi x)) + \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-y|} h\left(\frac{2}{\pi} \arctan y\right) dy \right| \\ &\leq \frac{1}{4} \|h\| + \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-y|} \|h\| dy, \end{aligned}$$

where $\|h\| = \sup_{t \in [-1, 1]} |h(t)|$. Using that (e.g. by substitution $t := y - x$ and symmetry)

$$\int_{-\infty}^{\infty} e^{-|x-y|} dy = \int_{-\infty}^{\infty} e^{-|t|} dt = 2 \int_0^{\infty} e^{-t} dt = 2,$$

we obtain

$$\|\Phi(f) - \Phi(g)\| = \sup_{x \in [-1, 1]} |\Phi(f)(x) - \Phi(g)(x)| \leq \frac{3}{4} \|f - g\|,$$

and thus Φ is a strict contraction since $3/4 < 1$.

6. a) Aiming to use the Implicit Function Theorem (ImFT), we define the constraint function $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$,

$$F(\mathbf{x}, \mathbf{u}) = F(x, y, u, v) := \begin{bmatrix} xye^{u+v} - 1 \\ x + y + u + av - 2 \end{bmatrix}$$

and will need to check that the matrix $\partial F/\partial \mathbf{u}$ is invertible. Namely, $F \in C^1$, and if $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u(x, y), v(x, y))$ is C^1 then, by differentiating the constraint $F(\mathbf{x}, \mathbf{u}(\mathbf{x})) = \mathbf{0}$ w.r.t. x and y , we obtain the matrix equation

$$\frac{\partial F}{\partial \mathbf{x}} + \frac{\partial F}{\partial \mathbf{u}} \frac{d\mathbf{u}}{d\mathbf{x}} = \mathbf{0}. \quad (1)$$

Furthermore, at $(x, y) = (1, 1)$ the constraint reads

$$\begin{cases} e^{u+v} = 1 \\ u + av = 0 \end{cases} \Leftrightarrow \begin{cases} u + v = 0 \\ u + av = 0 \end{cases} \quad (2)$$

Note that if $a = 1$ then there are several solutions (u, v) and we do not have a function graph at $(x, y) = (1, 1)$.

We also compute

$$\begin{aligned} \frac{\partial F}{\partial \mathbf{x}} &= \begin{bmatrix} ye^{u+v} & xe^{u+v} \\ 1 & 1 \end{bmatrix} = \{\text{at } x = 1, y = 1, u + v = 0\} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \\ \frac{\partial F}{\partial \mathbf{u}} &= \begin{bmatrix} xye^{u+v} & xye^{u+v} \\ 1 & a \end{bmatrix} = \{\text{at } x = 1, y = 1, u + v = 0\} = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}. \end{aligned}$$

The latter is invertible iff its determinant $a - 1$ is nonzero, i.e. $a \neq 1$. In this case

$$\left[\frac{\partial F}{\partial \mathbf{u}}(1, 1, u, v) \right]^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}^{-1} = \frac{1}{a-1} \begin{bmatrix} a & -1 \\ -1 & 1 \end{bmatrix}. \quad (3)$$

For $a \neq 1$ we have the unique solution $(u, v) = (0, 0)$ to (2) at $(x, y) = (1, 1)$.

Thus ImFT (Theorem 7.17) guarantees a solution $\mathbf{u}(\mathbf{x})$ which is C^1 in a neighborhood of $(x, y, u, v) = (1, 1, 0, 0)$.

- b) The linear (or affine) approximation to $\mathbf{f}(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ at $(x, y) = (1, 1)$ is given by the first two terms of the Taylor expansion

$$\mathbf{u}(1+h, 1+k) = \mathbf{u}(1, 1) + \frac{d\mathbf{u}}{d\mathbf{x}}(1, 1) \begin{bmatrix} h \\ k \end{bmatrix} + o(h, k).$$

By a) we have $a \neq 1$ and $(u(1, 1), v(1, 1)) = (0, 0)$, and furthermore, by (1) and (3)

$$\frac{d\mathbf{u}}{d\mathbf{x}}(1, 1) = - \left[\frac{\partial F}{\partial \mathbf{u}}(1, 1, 0, 0) \right]^{-1} \left[\frac{\partial F}{\partial \mathbf{x}}(1, 1, 0, 0) \right] = - \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}.$$

7. a) By the Riemann-Lebesgue theorem (Theorem 5.44), any function $f \in \mathcal{R}$ is bounded, so indeed $\mathcal{R} \subseteq \ell^\infty([a, b])$ where the latter is a Banach space with the norm $\|f\| = \sup_{x \in [a, b]} |f(x)|$. Now, assuming that $\mathcal{R} \ni f_n \rightarrow f \in \ell^\infty([a, b])$ uniformly on $[a, b]$ as $n \rightarrow \infty$, we have by Theorem 6.17 that $f \in \mathcal{R}$, and hence \mathcal{R} is a closed subset w.r.t. the uniform topology.

b) We may construct a sequence of functions $f_n \in \mathcal{R}$ on $[a, b]$ such that the pointwise limit $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists $\forall x \in [a, b]$ and is uniformly bounded, $f \in \ell^\infty([a, b])$, but $f \notin \mathcal{R}$. For example, enumerate $\mathbb{Q} \cap [a, b]$ as a sequence $(x_n)_{n \in \mathbb{N}}$ and take

$$f_n(x) := \begin{cases} 1, & \text{if } x \in \{x_0, x_1, \dots, x_n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [a, b], \\ 0, & \text{if } x \in [a, b] \setminus \mathbb{Q}, \end{cases}$$

so the pointwise limit f is the comb function which is not continuous anywhere on $[a, b]$ and therefore not Riemann integrable (see Exercise 5.13). Hence \mathcal{R} is not a closed subset w.r.t. the pointwise topology.

Note: One may also consider Rudin's example 7.4 with e.g. $m = n$, however it is hard to determine whether the pointwise limit exists.

8. Given (f_n) in $C^1(\mathbb{R})$ we consider the sequence of functions $(F_n : \mathbb{R} \rightarrow \mathbb{R})$,

$$F_n(x) := \frac{1}{n} \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq k \leq xn}} f_n(k/n), \quad x \in \mathbb{R}, \quad n = 1, 2, 3, \dots$$

restricted to $x \in [0, 1]$. However, it is helpful to recognize this as a Riemann sum approximation to an integral and thus first consider the sequence of functions (G_n) ,

$$G_n(x) := \int_0^x f_n(t) dt, \quad x \in \mathbb{R}, \quad n = 1, 2, 3, \dots$$

Namely, we claim that $f_n|_{[0,1]} \in C^1([0,1]) \subseteq \mathcal{R}$ are uniformly bounded, and thus by Proposition 6.28 (an application of Arzelà-Ascoli) there exists some subsequence $(f_{n'})$ such that $(G_{n'})$ converges uniformly on $[0, 1]$ (and by completeness the limit is also in $C([0, 1])$).

To prove our claim on boundedness consider the mean value theorem (Theorem 5.29),

$$f_n(x) = \underbrace{f_n(0)}_0 + x f_n'(t_x),$$

for $x \in [0, 1]$ and some $t_x \in (0, x)$. Therefore

$$\|f_n\|_{C([0,1])} \leq \|f_n'\|_{C([0,1])} \leq K < \infty \quad \forall n.$$

Next, let us consider the difference between the sequences (F_n) and (G_n) :

$$F_n(x) - G_n(x) = \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq k \leq xn-1}} \int_{k/n}^{(k+1)/n} (f_n(k/n) - f_n(t)) dt + \frac{1}{n} \sum_{\substack{k \in \mathbb{Z} \\ k=xn}} f_n(k/n) - \int_{\lfloor nx \rfloor/n}^x f_n(t) dt$$

The last two terms are bounded uniformly by $2\|f_n\|/n$, while for the remaining sum consisting of ca. xn integrals, again by the mean value theorem

$$f_n(t) = f_n(k/n) + (t - k/n)f'_n(t_k), \quad t, t_k \in [k/n, (k+1)/n],$$

so that by the box estimate

$$\int_{k/n}^{(k+1)/n} |f_n(k/n) - f_n(t)| dt \leq \frac{1}{n^2} \|f'_n\|.$$

Therefore, on $[0, 1]$,

$$\|F_n - G_n\| \leq \frac{1}{n} \|f'_n\| + \frac{2}{n} \|f_n\| \leq \frac{3K}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and thus the uniform convergence of the subsequence $(G_{n'})$ ensures that of the subsequence $(F_{n'})$ ($G_{n'}$ Cauchy implies $F_{n'}$ Cauchy).