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Matematiska institutionen
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Hemtentamen i matematik
Reell analys 1MA226
8 juni 2021

Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. Permitted aids: Course material (including books), lecture notes, old problems, exams and solutions.

1. Give examples or claim non-existence (with brief motivations) of:

a) A function $\mathbb{R}^2 \to \mathbb{R}$ which is discontinuous at every point of its domain $\mathbb{R}^2$.

b) A function $\mathbb{R}^2 \to \mathbb{R}$ which is differentiable but not continuously differentiable.

c) A function $\mathbb{R}^2 \to \mathbb{R}$ which is uniformly continuous but nowhere differentiable.

d) An equicontinuous and unbounded sequence of functions on $[0, 1]$.

e) A normed vector space in which the closed unit ball is neither separable, nor compact.

2. a) Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the sequence (x_n) defined by: $x_0 = 1$, $x_1 = -2$, and $x_{n+2} = x_{n+1}/x_n$.

b) Use lim sup and lim inf to define what it means for a function $f\colon \mathbb{R} \to \mathbb{R}$ to be continuous at a point $a \in \mathbb{R}$.

c) Could you also use these concepts for a function $f\colon \mathbb{Z} \to \mathbb{R}$ to define $f(\infty)$ and continuity at ∞?

3. On the set $\mathbb{Z}^2$ of integer points in the plane, denote $L = (0, 0)$ and define the distance function $d\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ by

$$d(P, Q) := \begin{cases} 0, & P = Q, \\ |P - L| + |Q - L|, & P \neq Q, \end{cases}$$

where $P, Q \in \mathbb{Z}^2$ and $|(x, y)| := \sqrt{x^2 + y^2}$ is the Euclidean metric on $\mathbb{R}^2$.

a) Show that $(\mathbb{Z}^2, d)$ defines a metric space.
(Amusingly, if L is London, d might be called the British Railway metric).

b) Does this metric space have the Heine-Borel property? Explain!

– Also see next page! / Var god vänd! –

4. Show that the series of functions

$$x \mapsto \sum_{n=1}^{\infty} \frac{x^n}{n} \sin(n\pi x)$$

converges pointwise on $[-1,1]$, uniformly on $(-1,1)$, and defines a continuously differentiable function on $(-1,1)$. Give an expression for its derivative on that interval.

5. Given two continuous functions

$$f_0 \colon [0,1] \to \mathbb{R}, \qquad g \colon [0,1] \to \mathbb{R},$$

define a sequence of functions $f_n \colon [0,1] \to \mathbb{R}$ iteratively for $n \in \mathbb{N}$ by

$$f_{n+1}(x) := \int_0^x \left(\frac{f_n(x)}{2} + g(x) \right) dx.$$

Show that (f_n) converges uniformly on $[0,1]$ to a continuously differentiable function f which satisfies

$$f'(x) = \frac{f(x)}{2} + g(x) \qquad \forall x \in (0,1).$$

Give expressions for $f(0)$ and $f(1)$.

6. Let $\mathbb{R}^{n\times n}$ denote the ring of real n by n matrices endowed with the operator norm, and consider the equation

$$ABA - BCB + CAC = I,$$

where $A, B, C \in \mathbb{R}^{n\times n}$ and I denotes the identity matrix. Show that this equation can be solved for A in terms of B and C locally near $(A,B,C) = (I,I,I)$, and give an expression for the linear approximation of that map $(B,C) \mapsto A$ at $(B,C) = (I,I)$. (Solving the problem with $n = 1$, i.e. variables in $\mathbb{R}$, is worth 2p.)

7. Assume that $f \in C([0,1])$ and find/compute the following limit (in terms of f):

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx.$$

8. Prove that there exists an increasing function $f \colon [0,1] \to \mathbb{R}$ which is discontinuous at every $x \in [0,1] \cap \mathbb{Q}$ and which is also Riemann integrable on $[0,1]$.

Good luck! / Lycka till!