UPPSALA UNIVERSITET
Matematiska institutionen
Douglas Lundholm

Hemtentamen i matematik
Reell analys 1MA226
8 june 2021

SOLUTION SUGGESTIONS.

1. a) We can take $f(x,y) = 1$ if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, and $f(x,y) = 0$ otherwise. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ (see Exercise 4.13), any ball contains points of both function values, which are separated by a distance 1.

b) Consider a function $g\colon \mathbb{R} \to \mathbb{R}$ which is continuously differentiable except at 0, where it is only differentiable. For example (see Example 5.13) $g(x) := x^2 \sin(1/x)$, $x \neq 0$, $g(0) := 0$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) := g(x)$. Then for all $(x,y) \in \mathbb{R}^2$

$$f(x+h, y+k) = g(x+h) = g(x) + g'(x)h + o(h) = f(x,y) + f'(x,y)\begin{bmatrix} h \\ k \end{bmatrix} + o(\sqrt{h^2+k^2}),$$

with $f'(x,y) = Df(x,y) = [g'(x), 0]$, which is in $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^{1\times 2}$ but does not vary continuously at $x = 0$:

$$f'(h,y) - f'(0,y) = [g'(h), 0] = [2h\sin(1/h) - \cos(1/h), 0] \nrightarrow [0,0] \quad \text{as } h \to 0.$$

c) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = W(x) + W(y),$$

where W denotes Weierstrass' monster, which is uniformly continuous $\mathbb{R} \to \mathbb{R}$, $|W(x) - W(y)| < \varepsilon$, if $|x-y| < \delta_\varepsilon$, but nowhere differentiable (see Theorem 6.16). Since, if $\sqrt{|h|^2 + |k|^2} < \delta_{\varepsilon/2}$,

$$|f(x+h, y+k) - f(x,y)| \le |W(x+h) - W(x)| + |W(y+k) - W(y)| < \varepsilon,$$

f is uniformly continuous on $\mathbb{R}^2$.

However, since the partial derivatives of f w.r.t. x respectively y coincide with the derivative of W, none of these exist and therefore f cannot be differentiable at any point $(x,y) \in \mathbb{R}^2$ (see Section 5.2.2 in the lecture notes).

d) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of equicontinuous functions on $[0,1]$ which is unbounded. Note that the functions f_n have to be bounded since they are continuous on a compact interval. We can e.g. take f_n to be a constant equal to n. Then $f_n(x) - f_n(y) = 0$ for all $x, y \in [0,1]$ and n, while $\|f_n\| \to \infty$ as $n \to \infty$.

e) By the proof of Theorem 4.69 we may consider the unit ball in ℓ^∞, the space of bounded real sequences with the supremum norm, which is not separable. By Exercise 4.41, this ball is not compact, because it contains sequences without subsequential limits (cp. Theorem 4.37); we may take the same family of unit vectors as a counterexample.

2. a) For the first few n we have

$$x_0 = 1,\ x_1 = -2,\ x_2 = -2,\ x_3 = 1,\ x_4 = -1/2,\ x_5 = -1/2,\ x_6 = 1,\ \text{etc.}$$

Note that in general

$$x_{n+3} = \frac{x_{n+2}}{x_{n+1}} = \frac{x_{n+1}}{x_n}\frac{1}{x_{n+1}} = (x_n)^{-1} \qquad \forall n \in \mathbb{N}.$$

Therefore, by induction, for all $k \in \mathbb{N}$,

$$\begin{aligned} x_{0+3k} &= (x_0)^{(-1)^k} = 1, \\ x_{1+3k} &= (x_1)^{(-1)^k} = (-2)^{(-1)^k}, \\ x_{2+3k} &= (x_2)^{(-1)^k} = (-2)^{(-1)^k}, \end{aligned}$$

the latter being equal to -2 for even k and $-1/2$ for odd k. Since (x_n) takes then only three distinct values, $\{-2, -1/2, 1\}$, and each of these infinitely many times, for any subsequence $(x_{n'})$ to have a limit it needs to eventually converge to one of these values. Among those limits, -2 is smallest and 1 largest. Thus we conclude that

$$\liminf_{n\to\infty} x_n = -2, \qquad \limsup_{n\to\infty} x_n = 1.$$

b) Continuity of f at a demands that

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x).$$

Furthermore, for these left and right-hand limits to exist, it is required that for any sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $x_n \to a^-$ (respectively $x_n \to a^+$) as $n \to \infty$, i.e. such that $x_n < a$ (respectively $x_n > a$) and the limit exists

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = a,$$

we demand that $f(x_n) \to f(a)$ as $n \to \infty$, i.e. the limit exists

$$\liminf_{n\to\infty} f(x_n) = \limsup_{n\to\infty} f(x_n) = f(a).$$

The limits can also be treated from both sides as a single one, i.e. $(x_n \neq a)$

$$\lim_{x\to a} f(x) = \liminf_{n\to\infty} f(x_n) = \limsup_{n\to\infty} f(x_n) = f(a).$$

c) With respect to the integers $\mathbb{Z}$, we may think of the limit point ∞ at infinity either as the one-sided limit $\mathbb{Z} \ni n \to +\infty$, or as the two-sided limit $\mathbb{Z} \ni n \to \pm\infty$. In either case we may define $f(\infty)$ as the corresponding limit of the sequence of function values $(f(n))_{n\in\pm\mathbb{N}}$ if it exists, i.e. if

$$\liminf_{n\to\pm\infty} f(n) = \limsup_{n\to\pm\infty} f(n).$$

In case all the considered limits agree, this defines $f(\infty)$. This also holds in the extended sense if these limits (limsup and liminf) are either $-\infty$ or $+\infty$.

3. a) We verify the axioms of a metric (Definition 4.9): $d\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}_+$, and

(i): $d(P,Q) = |P| + |Q| = |Q| + |P| = d(Q,P)$ for any $P \neq Q \in \mathbb{Z}^2$.

(ii): $d(P,R) = |P| + |R| \leq |P| + |Q| + |Q| + |R| = d(P,Q) + d(Q,R)$ for any distinct $P, Q, R \in \mathbb{Z}^2$. If any two of the points are equal, $d(P,R) \leq d(P,Q) + d(Q,R)$ trivially holds.

(iii): $d(P,P) = 0$ for any $P \in \mathbb{Z}^2$, and if $P \neq Q$ then $d(P,Q) = |P| + |Q| > 0$ since either P or Q is not $L = (0,0)$, and by the property of the Euclidean **norm**.

b) Yes, it has the Heine-Borel property, that is every bounded and closed subset is compact (Definition 4.34). Since any two distinct points are separated by a finite distance 1, the space is discrete, and in any discrete space every subset is closed (since it is complement to a union of singleton open sets $\{x\}$). Furthermore, any bounded subset B is finite, since if it would be infinite, the set $\{|P - L| : P \in B\}$ would be unbounded. Lastly, any finite set is compact since any covering by subsets of $\mathbb{Z}^2$ may be reduced to a finite subcovering (in fact by the discreteness of the space it is at most non-trivially covered by as many sets as it has elements).

4. Note that there was an error in the problem formulation: add "compact subsets of" after "uniformly on" (checking whether convergence is uniform on $[-1,1]$ is much harder).

We write for convenience $F(x) = \sum_{n=1}^{\infty} f_n(x)$ where

$$f_n(x) := \frac{1}{n} x^n \sin(n\pi x), \qquad n \in \mathbb{N}^+,\ x \in \mathbb{R}.$$

Note that $f_n \in C(\mathbb{R})$ but is not bounded, $\|f_n\|_{C(\mathbb{R})} = \infty$. However on the interval $I_\delta := [-\delta, \delta]$, $\delta > 0$, each f_n is bounded in the norm $\|f\| := \|f\|_{C(I_\delta)} = \sup_{-\delta \leq x \leq \delta} |f(x)|$:

$$\|f_n\| \leq \delta^n / n.$$

Now fix $0 < \delta < 1$. By the convergence of the series

$$\sum_{n=0}^{\infty} \delta^n = (1 - \delta)^{-1},$$

we can estimate F uniformly on $C(I_\delta)$

$$\|F\| \leq \sum_{n=1}^{\infty} \|f_n\| \leq (1 - \delta)^{-1}.$$

Further, by the majorization theorem for series (Theorem 4.66) and the completeness of the space $C(I_\delta)$, the sequence $F_N = \sum_{n=1}^N f_n$ converges uniformly to the pointwise limit F as $N \to \infty$. We also note that the series converges pointwise at $x = \pm 1$ since every term is zero in that case.

Let us now compute the derivative of the finite partial sum F_N:

$$F_N'(x) = \sum_{n=1}^{N} \left(x^{n-1} \sin(n\pi x) + x^n \pi \cos(n\pi x) \right).$$

Each term of this sum of functions is $f_n' \in C(\mathbb{R})$, for which

$$\|f_n'\| \le \delta^{n-1}(1 + \delta\pi) \qquad \text{on } I_\delta.$$

Thus, we obtain the uniform bound

$$\|F_N'\| \le \sum_{n=1}^{\infty} \|f_n'\| \le (1-\delta)^{-1}(1+\delta\pi),$$

and again we conclude by majorization that the sequence (F_N') converges uniformly on I_δ to its pointwise limit.

By Theorems 6.8/9 and 6.14 we conclude that, uniformly $F_N \to F \in C^1(I_\delta)$, and

$$F' = \lim_{N\to\infty} F_N' = \sum_{n=1}^{\infty} f_n' = \sum_{n=1}^{\infty} \left(x^{n-1} \sin(n\pi x) + x^n \pi \cos(n\pi x) \right).$$

Furthermore, since $\delta < 1$ was arbitrary we obtain this formula pointwise on $I_1 = (-1, 1)$, and also $F \in C^1(I_1)$. (However, note that the series does not converge at $x = \pm 1$.)

5. We may consider a map Φ on $C([0,1])$,

$$\Phi(f)(x) := \int_0^x \left(\frac{1}{2} f + g \right), \qquad x \in [0,1],$$

that we iterate in our definition of f_n,

$$f_{n+1} := \Phi(f_n), \qquad n = 0, 1, 2, \ldots$$

Note that since continuity on $[0,1]$ implies Riemann integrability, by the Fundamental Theorem of Integral Calculus (FTIC; see e.g. Theorem 5.46), the image $\Phi(f)$ is in $C([0,1])$ and is even differentiable,

$$\Phi(f)'(x) = \frac{1}{2} f(x) + g(x).$$

Further, we note that

$$\|\Phi(f) - \Phi(h)\| = \sup_{x\in[0,1]} \left| \int_0^x \frac{1}{2}(f-h) \right| \le \frac{1}{2}\|f-h\|,$$

by the triangle/box inequality for integrals.

Therefore $\Phi\colon C([-1,1]) \to C([-1,1])$ is a strict contraction on a Banach space, and we may use Banach's Fixpoint Theorem (BFT) to conclude that Φ has a unique fixpoint,

$$f = \Phi(f), \qquad f \in C([0,1]),$$

to which (f_n) converges uniformly. It also follows by the FTIC that f is differentiable, satisfying

$$f'(x) = \Phi(f)'(x) = \frac{1}{2}f(x) + g(x),$$

and furthermore continuously differentiable since the r.h.s. is continuous.

By the definition of f_n we have

$$f(0) = \lim_{n\to\infty} f_n(0) = 0$$

and, by the convergence of definite integrals (Theorem 6.17),

$$f(1) = \lim_{n\to\infty} f_n(1) = \int_0^1 \left(\frac{1}{2}f + g\right).$$

OPTIONAL: We may actually solve the problem completely: By plugging in the ansatz $f(x) = e^{x/2}G(x)$ into the differential equation, where G is continuously differentiable (since f is, by the above result), we obtain

$$0 = f'(x) - \frac{1}{2}f(x) - g(x) = e^{x/2}G'(x) - g(x), \qquad \Rightarrow \qquad G'(x) = e^{-x/2}g(x)$$

i.e.

$$G(x) = \int_0^x e^{-t/2}g(t)\,dt + G(0),$$

where $G(0) = f(0) = 0$. It follows that $f(1) = e^{1/2}\int_0^1 e^{-t/2}g(t)\,dt$.

6. First note that $\mathbb{R}^{n\times n}$ with the norm $\|\cdot\| = \|\cdot\|_{\mathrm{op}}$ is a Banach ring (see sections 4.10 and 5.2 of the lecture notes), and furthermore, viewed as a vector space of matrices it may be equivalently treated as $\mathbb{R}^{n^2}$ with the Euclidean norm, since any two norms in finite dimensions are equivalent (see Exercise 4.37). Thus, conceptually the problem simplifies to one on Euclidean space $\mathbb{R}^N$, and one may treat the matrix components as variables while keeping in mind that the norms may differ by constants. For convenience we will deal directly with matrices and use the multiplicative property of the operator norm.

Aiming to use the Implicit Function Theorem (ImFT), we define the constraint function

$$F\colon \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n},$$

$$F(A,B,C):=ABA-BCB+CAC-I.$$

We note that indeed $F(I,I,I)=0$, so certainly $A=I,B=I,C=I$ solves the equation. We look more generally for a function

$$A\colon \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n},$$

$$(B,C)\mapsto A(B,C),$$

such that

$$F(A(B,C),B,C)=0,$$

in a neighborhood of the point $(B,C)=(I,I)$.

Let us first verify that F is continuously differentiable. Namely, for any $A,B,C,H,K,L\in\mathbb{R}^{n\times n}$ we compute

$$\begin{aligned}
&F(A+H,B+K,C+L)\\
&=(A+H)(B+K)(A+H)-(B+K)(C+L)(B+K)+(C+L)(A+H)(C+L)-I\\
&=ABA-BCB+CAC-I\\
&\quad+HBA+AKA+ABH-KCB-BLB-BCK+LAC+CHC+CAL\\
&\quad+O(\|H\|^2+\|K\|^2+\|L\|^2)\\
&=F(A,B,C)+F'(A,B,C)[H,K,L]+o(H,K,L),
\end{aligned}$$

with the (continuous) linear map $F'(A,B,C)\in\mathrm{Hom}(\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$

$$F'(A,B,C)[H,K,L]:=\frac{\partial F}{\partial A}(A,B,C)[H]+\frac{\partial F}{\partial B}(A,B,C)[K]+\frac{\partial F}{\partial C}(A,B,C)[L],$$

consisting of the partial derivatives (also continuous linear maps, in $\mathrm{Hom}(\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$)

$$\begin{aligned}
\frac{\partial F}{\partial A}(A,B,C)[H]&:=HBA+ABH+CHC,\\
\frac{\partial F}{\partial B}(A,B,C)[K]&:=AKA-KCB-BCK,\\
\frac{\partial F}{\partial C}(A,B,C)[L]&:=-BLB+LAC+CAL.
\end{aligned}$$

We have estimated all the higher-order terms using the norm, such as

$$\|AKH\|\le\|A\|\,\|K\|\,\|H\|\le\frac{\|A\|}{2}(\|H\|^2+\|K\|^2+\|L\|^2)=o(H,K,L).$$

Note that all the partial derivatives are continuous in A, B and C, since for example

$$\begin{aligned}
&\frac{\partial F}{\partial A}(A+H', B+K', C+L')[H] - \frac{\partial F}{\partial A}(A,B,C)[H] \\
&= H(B+K')(A+H') + (A+H')(B+K')H + (C+L')H(C+L') - HBA - ABH - CHC \\
&= O(H', K', L')\,\|H\|\,,
\end{aligned}$$

so

$$\left\| \frac{\partial F}{\partial A}(A+H', B+K', C+L') - \frac{\partial F}{\partial A}(A,B,C) \right\| = O(H', K', L'),$$

and therefore $F \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}; \mathbb{R}^{n\times n})$.
At the point $(A,B,C) = (I,I,I)$ we have

$$\frac{\partial F}{\partial A}(I,I,I)[H] = 3H, \qquad \frac{\partial F}{\partial B}(I,I,I)[K] = -K, \qquad \frac{\partial F}{\partial C}(I,I,I)[L] = L,$$

To derive the condition for the validity of ImFT (completely analogous to its standard formulation on $\mathbb{R}^{N\times M} \to \mathbb{R}^N$) we make the assumption that $A = A(B,C)$ is C^1 and plug a small perturbation of B and C around I into the equation $F(A,B,C) = 0$:

$$\begin{aligned}
0 &= F\big(A(I+K, I+L), I+K, I+L\big) \\
&= F\left(\underbrace{A(I,I)}_{I} + \frac{\partial A}{\partial B}(I,I)[K] + \frac{\partial A}{\partial C}(I,I)[L] + o(K,L),\ I+K,\ I+L\right) \\
&= \underbrace{F(I,I,I)}_{0} + \frac{\partial F}{\partial A}(I,I,I)\left[\frac{\partial A}{\partial B}(I,I)[K] + \frac{\partial A}{\partial C}(I,I)[L] + o(K,L)\right] \\
&\quad + \frac{\partial F}{\partial B}(I,I,I)[K] + \frac{\partial F}{\partial C}(I,I,I)[L] + o(K,L).
\end{aligned}$$

Hence, taking $K \to 0$, $L = 0$ respectively $K = 0$, $L \to 0$, we obtain the linear equations

$$\begin{aligned}
0 &= \frac{\partial F}{\partial A}(I,I,I)\left[\frac{\partial A}{\partial B}(I,I)[K]\right] + \frac{\partial F}{\partial B}(I,I,I)[K], \\
0 &= \frac{\partial F}{\partial A}(I,I,I)\left[\frac{\partial A}{\partial C}(I,I)[L]\right] + \frac{\partial F}{\partial C}(I,I,I)[L],
\end{aligned}$$

which may be solved for $\partial A/\partial B$, $\partial A/\partial C$ if the matrix $F_A := \frac{\partial F}{\partial A}(I,I,I)$ is invertible, which is indeed the case, with $F_A = 3I$ and $F_A^{-1} = \frac{1}{3}I$. We conclude then that

$$\begin{aligned}
\frac{\partial A}{\partial B}(I,I) &= -F_A^{-1}\frac{\partial F}{\partial B}(I,I,I) = \frac{1}{3}I, \\
\frac{\partial A}{\partial C}(I,I) &= -F_A^{-1}\frac{\partial F}{\partial C}(I,I,I) = -\frac{1}{3}I,
\end{aligned}$$

and thus obtain the linear (affine) approximation

$$A(I+K, I+L) = I + \frac{1}{3}K - \frac{1}{3}L + o(K, L).$$

Again, by mapping the problem to $\mathbb{R}^N$ and using Theorem 7.17 and Remark 7.18, or using Theorem 7.19 directly, invertibility of F_A at (I, I) guarantees the solvability of $F(A, B, C) = 0$ for $A = A(B, C)$ which is C^1 in some neighborhood of (I, I).

We may also perform the following quick check for the correctness of our computations:

$$0 = F\big(I+H, I+K, I+L\big) = 0 + 3H - K + L + o(H, K, L),$$

so that $H = \frac{1}{3}K - \frac{1}{3}L + o(H, K, L)$ and

$$A(I+K, I+L) =: I + H = I + \frac{1}{3}K - \frac{1}{3}L + o(K, L).$$

(In the case that $n = 1$ we have $F\colon \mathbb{R}^3 \to \mathbb{R}$ with the usual Euclidean spaces and partial derivatives, and thus a more direct application of ImFT.)

7. We might first try to use partial integration (Rudin 6.22),

$$\int_0^1 x^n F'(x)\,dx = [x^n F(x)]_{x=0}^1 - \int_0^1 n x^{n-1} F(x)\,dx,$$

with $F(x) := \int_0^x f(t)\,dt$ continuously differentiable, or

$$\int_0^1 x^n f(x)\,dx = \left[\frac{x^{n+1}}{n+1} f(x)\right]_{x=0}^1 - \int_0^1 \frac{x^{n+1}}{n+1} f'(x)\,dx.$$

The latter looks very promising, however we don't know that f is differentiable and thus that we can bound uniformly and take limits. The former approach will also be problematic when we take the given limit.

Instead, let us use that $nx^n \to 0$ pointwise on $[0, 1)$ and the continuity of f at $x = 1$: For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(1) - \varepsilon \le f(x) \le f(1) + \varepsilon \qquad \forall 1 - \delta < x \le 1.$$

Thus we split the integral

$$\int_0^1 n x^n f(x)\,dx = \int_0^{1-\delta} n x^n f(x)\,dx + \int_{1-\delta}^1 n x^n f(x)\,dx.$$

For the first part we estimate using the supremum norm of f (bounded on compact),

$$\left|\int_0^{1-\delta} n x^n f(x)\,dx\right| \le \int_0^{1-\delta} n x^n dx\,\|f\| = \frac{n}{n+1}(1-\delta)^{n+1}\,\|f\| \to 0,$$

as $n \to \infty$, and for the second part

$$\int_{1-\delta}^{1} nx^n f(x)\,dx = \int_{1-\delta}^{1} nx^n \left(f(x) - f(1)\right) dx + \int_{1-\delta}^{1} nx^n f(1)\,dx,$$

where

$$\left|\int_{1-\delta}^{1} nx^n \left(f(x) - f(1)\right) dx\right| \le \varepsilon \int_{1-\delta}^{1} nx^n dx = \varepsilon \frac{n}{n+1}\left(1 - (1-\delta)^{n+1}\right) \to \varepsilon,$$

and similarly for the remaining integral

$$\int_{1-\delta}^{1} nx^n f(1)\,dx \to f(1), \qquad n \to \infty.$$

Since $\varepsilon > 0$ was arbitrary we must have that the limit exists and equals $f(1)$:

$$\left|\int_{0}^{1} nx^n f(x)\,dx - f(1)\right| \le 4\varepsilon,$$

for $n \ge N(\varepsilon)$ large enough (more precisely, we used here the triangle inequality and kept one ε for margin in each of the three limits above).

8. Using that $\mathbb{Q} \cap [0,1]$ is countable we may enumerate this set as $\{q_n\}_{n\in\mathbb{N}^+}$.

Define the simple step function

$$\chi_{(q_n,1]}(x) := \begin{cases} 0, & x \le q_n, \\ 1, & x > q_n, \end{cases}$$

and the series $f\colon [0,1] \to \mathbb{R}_+$,

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{(q_n,1]}(x).$$

We claim that this function f has the desired properties.

For convenience let us denote the partial sums

$$f_N := \sum_{n=1}^{N} \frac{1}{2^n} \chi_{(q_n,1]}.$$

Note that f_N is pointwise monotonously increasing in N, and is uniformly bounded,

$$\|f_N\| \le \sum_{n=1}^{N} \frac{1}{2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Hence, since bounded and increasing sequences in $\mathbb{R}$ converge (Theorem 3.11), f_N converges pointwise to f. Furthermore, we have for any $x \in [0,1]$, $N \geq M \geq 1$

$$|f_N(x) - f_M(x)| \leq \sum_{n=M}^{N-1} \frac{1}{2^n} \leq \sum_{n=M}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^{M-1}},$$

and thus (f_N) is uniformly Cauchy, and $f_N \to f$ uniformly on $[0,1]$.

Let's check that f is increasing. In fact, limits of increasing functions are increasing: for $1 \geq y \geq x \geq 0$

$$f(y) - f(x) = \lim_{N\to\infty} (f_N(y) - f_N(x)) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} \left(\chi_{(q_n,1]}(y) - \chi_{(q_n,1]}(x)\right) \geq 0.$$

Next, consider its discontinuities, which are expected at each q_n. Similarly as above, the function $f - \frac{1}{2^n}\chi_{(q_n,1]}$ is increasing for any n. Hence

$$\lim_{x\to q_n^-} f(x) = \lim_{x\to q_n^-} \left(f(x) - \frac{1}{2^n}\chi_{(q_n,1]}(x)\right) \leq \lim_{x\to q_n^+} \left(f(x) - \frac{1}{2^n}\chi_{(q_n,1]}(x)\right) = \lim_{x\to q_n^+} f(x) - \frac{1}{2^n},$$

and therefore

$$\lim_{x\to q_n^+} f(x) - \lim_{x\to q_n^-} f(x) \geq \frac{1}{2^n}.$$

(Note that f is even strictly increasing since for any two points $0 \leq x < y \leq 1$ there will eventually be an n s.t. $x < q_n < y$.)

Finally, let's check that f is Riemann integrable. Either we use the Riemann-Lebesgue theorem (Theorem 5.45) with the fact that f is bounded and its set of discontinuities is a zero set, or more explicitly we may use Theorem 6.17 for the convergence of integrals. Namely, $f_N \in \mathscr{R}$ since it is a stair function, i.e. constant on each interval

$$(0, q_{1'}),\ (q_{2'}, q_{3'}),\ (q_{3'}, q_{4'}), \ldots, (q_{N'}, 1),$$

where we ordered the points $0 \leq q_{1'} < \ldots < q_{N'} \leq 1$. Taking any $\varepsilon > 0$ there exists N such that

$$\|f - f_N\| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon,$$

thus

$$f_N(x) \leq f(x) \leq f_N(x) + \varepsilon \qquad \forall x \in [0,1],$$

and we obtain, for any partition P of $[0,1]$ which is finer than that defined by the points $\{q_n\}_{n=1}^N$, the upper and lower integrals/sums

$$0 \leq \overline{\int} f - \underline{\int} f \leq U(f,P) - L(f,P) \leq U(f_N + \varepsilon, P) - L(f_N, P) \leq \varepsilon.$$

Since ε was arbitrary we obtain $f \in \mathscr{R}$ and in fact

$$\int_0^1 f = \lim_{N\to\infty} \int_0^1 f_N = \sum_{n=1}^{\infty} \frac{1 - q_n}{2^n}.$$