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Matematiska institutionen
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Hemtentamen i matematik
Reell analys 1MA226
23 augusti 2021

Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. Permitted aids: Course material (including books), lecture notes, old problems, exams and solutions.

1. Give examples or claim non-existence (with brief motivations) of:

a) A differentiable function $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with discontinuous partial derivatives.
b) A linear function between normed vector spaces which is not differentiable.
c) A set of functions with strictly greater cardinality than that of $\mathbb{R}$.
d) A bounded sequence of continuous functions on $[0,1]$ which converges pointwise but does not have a uniformly convergent subsequence.
e) A continuous function $\mathbb{Q} \to \mathbb{R}$ which does not have the intermediate value property.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the sequences (x_n) defined by:

a) $x_n = \sum_{k=1}^{n} k \sin \frac{k\pi}{2}$ for $n \geq 1$.
b) $x_1 = 2$ and $x_{n+1} = \left(x_n^{1/n} - \frac{1}{n(n+1)}\right)^{n+1}$ for $n \geq 1$.
c) Do the above sequences contain convergent subsequences?

3. Prove that the only subsets of $\mathbb{R}^n$ (with its usual metric topology) that are both open and closed are the empty set $\emptyset$ and the whole set $\mathbb{R}^n$.

4. Show that the series of functions

$$x \mapsto \sum_{n=1}^{\infty} \int_0^x \frac{t^n}{n!} \cos t \, dt$$

converges pointwise on $\mathbb{R}$, uniformly on compact subsets of $\mathbb{R}$, and defines a continuously differentiable function on $\mathbb{R}$. Compute its derivative.

– Also see next page! / Var god vänd! –

5. Let $a \leq b$. Prove, or disprove, that the space

$$M = \left\{ f \in C([a,b]) : |f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for every } x, y \in [a,b],\ f(a) = 0 \right\}$$

is a compact space under the metric $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$.

6. Let $L = \mathbb{R}^{n \times n}$ denote the ring of real n by n matrices endowed with the operator norm. Define $f\colon L \to L$ by $f(T) = \exp(T(I - T))$, where $\exp\colon L \to L$ is the exponential function defined by its usual power series expansion.
Is f locally invertible near the identity matrix I?

(Solving the problem with $n = 1$, i.e. $L = \mathbb{R}$ and $I = 1$, is worth 2p. Also, you may use that exp is continuously differentiable for all n.)

7. a) Compute the upper and lower integrals of the function $f\colon [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 2 - x, & x \in [0,1] \setminus \mathbb{Q}, \\ x, & x \in \mathbb{Q} \cap [0,1], \end{cases}$$

and conclude that it is not Riemann integrable.

b) Is it possible to change the definition of f on a null set (a set coverable by an arbitrarily thin collection of intervals) such that it becomes Riemann integrable, and if so, what is its resulting integral?

8. Let $g_n\colon [0,1] \to \mathbb{R}$, $n = 0, 1, 2, \ldots$, be a sequence of differentiable functions such that the sequence $g_n'\colon [0,1] \to \mathbb{R}$ is uniformly bounded.

a) Show that there is a sequence of constants $c_n \in \mathbb{R}$ such that the sequence of functions $h_n(x) = g_n(x) - c_n$ on $[0,1]$ has a uniformly convergent subsequence.

b) Show that if $\int_0^1 g_n$ is a bounded sequence in $\mathbb{R}$ then the sequence g_n has a uniformly convergent subsequence.

(You may use the result of a) in b) freely.)

Good luck! / Lycka till!