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SOLUTION SUGGESTIONS. (Unless stated, referenced theorems are from the lecture notes.)

1. a) We can take

$$\mathbf{F}(x,y) = \begin{cases} \big((x^2+y^2)\sin(1/\sqrt{x^2+y^2}), (x^2+y^2)\sin(1/\sqrt{x^2+y^2})\big), & (x,y) \neq \mathbf{0} \\ \mathbf{0}, & (x,y) = \mathbf{0}, \end{cases}$$

which is smooth outside the origin $\mathbf{0} = (0,0)$, hence its partial derivatives are continuous there. Further, by the boundedness of sin,

$$\mathbf{F}(0+h, 0+k) = (h^2+k^2)\sin(1/\sqrt{h^2+k^2})\big(1,1\big) = \mathbf{F}(\mathbf{0}) + \mathbf{0} + o(h,k),$$

therefore it is differentiable also at $\mathbf{0}$, with derivative zero. Thus, for example, $\frac{\partial F_1}{\partial x}(\mathbf{0}) = 0$, however

$$\frac{\partial F_1}{\partial x}(h,0) = 2h\sin(1/h) - \cos(1/h) \nrightarrow 0, \quad h \to 0^+,$$

so that $\frac{\partial F_1}{\partial x}$ is not continuous at $\mathbf{0}$. The same conclusion holds for $\frac{\partial F_1}{\partial y}$, $\frac{\partial F_2}{\partial x}$ and $\frac{\partial F_2}{\partial y}$.

b) By definition of differentiability (see Definition 5.2), any linear function $L\colon V \to W$,

$$L[x+h] = L[x] + L[h] + 0 \qquad \forall x,h \in V,$$

is differentiable iff $L \in \mathrm{Hom(V,W)}$, i.e. if it is bounded/continuous, i.e. $\|L\|_{\mathrm{op}} < \infty$. Further, in finite dimensions every linear function is bounded/continuous (see Exercise 5.5), so we need to consider infinite dimensions (for V). An example is given in Example 5.11, where $V = \ell^\infty_c$ (compactly supported sequences), $W = \mathbb{R}$, and

$$L[x] = \sum_{n=0}^{\infty} x_n.$$

c) The set of all functions $f\colon \mathcal{P}(\mathbb{R}) \to \{0\}$ are in bijection to the power set $\mathcal{P}(\mathbb{R})$ which, by Cantor's Theorem, has strictly greater cardinality than $\mathbb{R}$.

d) We can take the sequence $(f_n)_{n\in\mathbb{N}^+}$ from Example 6.12,

$$f_n(x) = \begin{cases} 2nx, & 0 \le x \le 1/(2n), \\ 2-2nx, & 1/(2n) < x \le 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

which tends pointwise to zero, so that any uniformly convergent subsequence must tend to zero as well. However $\|f_n\| = 1 \nrightarrow 0$ as $n \to \infty$.

We might also consider the sequence $g_n(x) = x^n$, which converges pointwise to a discontinuous function.

e) Consider the function $f\colon \mathbb{Q} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & x < \sqrt{2}, \\ 1, & x > \sqrt{2}. \end{cases}$$

It does not take any intermediate values $y \in (0,1)$ however it is continuous at every point x of its domain, since either $x < \sqrt{2}$ or $x > \sqrt{2}$.

2. a) Since

$$\sin \frac{k\pi}{2} = \begin{cases} 1, & \text{if } k = 4m+1, \\ -1, & \text{if } k = 4m+3, \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

(where $m \in \mathbb{N}$), for the first few n we have

$$x_1 = 1,\ x_2 = 1,\ x_3 = -2,\ x_4 = -2,\ x_5 = 3,\ x_6 = 3,\ x_7 = -4,\ \text{etc.},$$

and in general for $m \geq 1$

$$x_{4m+2} = x_{4m+1} = x_{4m} + 4m + 1,$$

$$x_{4(m+1)} = x_{4m+3} = x_{4m+2} - 4m - 3 = x_{4m} - 2 = \ldots = -2(m+1),$$

so

$$x_{4m+1} = x_{4m+2} = 2m+1, \qquad x_{4m+3} = x_{4m+4} = -2(m+1).$$

Every subsequence either keeps its sign eventually and diverges to $\pm\infty$ or switches sign indefinitely and then simply diverges. Thus we conclude that

$$\liminf_{n\to\infty} x_n = -\infty, \qquad \limsup_{n\to\infty} x_n = +\infty.$$

b) Let us try to separate expressions of n and $n+1$. We observe that

$$x_{n+1}^{\frac{1}{n+1}} - x_n^{\frac{1}{n}} = -\frac{1}{n(n+1)} = \frac{1}{n+1} - \frac{1}{n}$$

so that the combination

$$C(n) := x_n^{\frac{1}{n}} - \frac{1}{n} = x_{n+1}^{\frac{1}{n+1}} - \frac{1}{n+1} = C(n+1)$$

is actually independent of $n \Rightarrow C(n) = C(1) = 1$. Thus,

$$x_n = \left(C(n) + \frac{1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

which converges to e. Hence, $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n = e$.

c) In a) no, because every subsequence is divergent.
In b) yes, every subsequence converges since $\lim_{n\to\infty} x_n = e$.

3. We first note that $\emptyset$ and $\mathbb{R}^n$ are each others' complements, and are both open and closed by any of our definitions of these concepts. Next, assume that $A \subsetneq \mathbb{R}^n$ is a nonempty subset which is both open and closed. Then its complement $A^c = \mathbb{R}^n \setminus A$ is also a nonempty subset which is both open and closed. Hence, we have a disjoint union $\mathbb{R}^n = A \sqcup A^c$ or, if we prefer, a coloring of each of the points of $\mathbb{R}^n$ into two colors, say red and blue. Now take one point $x \in A$ (red) and one point $y \in A^c$ (blue). Then consider the line segment

$$[x, y] := \{z \in \mathbb{R}^n : \exists t \in [0, 1] \text{ s.t. } z = x + t(y - x)\},$$

and its middle point $p = (x + y)/2$. Now, either $p \in A$ or $p \in A^c$, i.e. red or blue. If it is red, let us consider the segment $[p, y]$, and if it is blue consider $[x, p]$, i.e. keep the half-segment where the endpoints have different colors. Now, iterate this procedure on that half-segment, etc., producing two sequences of endpoints $x_0, x_1, \ldots$ in A and $y_0, y_1, \ldots$ in A^c, which satisfy

$$|x_k - y_k| \le 2^{-k}|x - y|, \quad \forall k \in \{0, 1, 2, \ldots\}.$$

Furthermore, they are all on the closed segment $[x, y]$ and eventually in $[x_N, y_N] \subseteq [x_{N-1}, y_{N-1}]$ for any $N \ge 1$. Similarly to the proofs of Bolzano-Weierstrass, Heine-Borel, or Lemma 4.50 (Ball enclosure), we obtain

$$x_k \text{ and } y_k \xrightarrow{k\to\infty} \bigcap_{N=0}^{\infty} [x_N, y_N] =: q \in \mathbb{R}^n.$$

Since A is closed, $q \in A$, but also, since A^c is closed, $q \in A^c$, that is, this intersection or boundary point q is both red and blue, which is impossible, $A \cap A^c = \emptyset$.

4. Let us define for convenience the partial sums $F_N := \sum_{n=1}^{N} f_n$, where

$$f_n(x) := \int_0^x g_n(t)\, dt, \qquad g_n(x) := \frac{x^n}{n!} \cos x, \qquad n \in \mathbb{N}^+, \ x \in \mathbb{R}.$$

The integrand g_n is smooth. Pointwise, $x \in \mathbb{R}$, we have by the box bound for integrals (and with the usual norm $\|f\|_{C(X)} := \sup_{x\in X} |f(x)|$)

$$|F_N(x)| \le \sum_{n=1}^{N} |f_n(x)|, \qquad |f_n(x)| \le |x| \|g_n\|_{C([-x,x])} \le |x| \frac{|x|^n}{n!},$$

and thus by the majorization theorem for series (Theorem 4.66), i.e. the absolute convergence

$$\sum_{n=1}^{\infty} |f_n(x)| \leq |x| \sum_{n=0}^{\infty} \frac{|x|^n}{n!} = |x|e^{|x|},$$

we obtain pointwise convergence $F_N(x) \to F(x) := \sum_{n=1}^{\infty} f_n(x)$ as $N \to \infty$.

On any compact subset $K \subseteq [-R, R] \subseteq \mathbb{R}$,

$$\sum_{n=1}^{\infty} \|f_n\|_{C(K)} \leq Re^R,$$

and thus by the majorization theorem in $C([-R, R])$ and Theorems 6.8/9, F_N is uniformly Cauchy and converges uniformly in $C([-R, R])$ to F, and thus also in $C(K)$. (But not in $C(\mathbb{R})$.)

Further, we have by finite summation of integrals

$$F_N(x) = \sum_{n=1}^{N} \int_0^x g_n(t)\, dt = \int_0^x \sum_{n=1}^{N} g_n(t)\, dt,$$

and in the integrand, again by the majorization theorem in $C([-x, x])$, the uniform convergence of Riemann-integrable (actually smooth) functions

$$C([-x, x]) \ni G_N := \sum_{n=1}^{N} g_n \to G, \quad N \to \infty,$$

to a Riemann-integrable (actually smooth) function

$$G(t) := \sum_{n=1}^{\infty} g_n(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cos t - \cos t = (e^t - 1)\cos t,$$

so that by Theorem 6.17

$$F(x) = \lim_{N\to\infty} \int_0^x G_N = \int_0^x G.$$

Therefore, by the fundamental theorem of integral calculus, F is continuously differentiable, on all of $\mathbb{R}$, with

$$F'(x) = G(x) = (e^x - 1)\cos x.$$

5. In the case that $a = b$ we only have one point and a single function, $M = \{0\}$, which is trivially compact. Assume now that $a < b$. The metric d is our usual metric on $C([a, b])$, so M is a subset of the standard metric space $(C([a, b]), d)$. By Arzela-Ascoli, such a subset is compact if and only if it is closed, bounded and equicontinuous. Let us check these conditions one at a time.

Since for all $f \in M$ and $x \in [a, b]$

$$|f(x)| = |f(x) - f(a)| \leq \sqrt{|x-a|} \leq \sqrt{|b-a|},$$

M is certainly uniformly **bounded**.

Since for any $\varepsilon > 0$ there is $\delta = \varepsilon^2$ such that for any $f \in M$ and any $x, y \in [a, b]$ s.t. $|x - y| < \delta$

$$|f(x) - f(y)| \leq \sqrt{|x-y|} < \sqrt{\delta} = \varepsilon,$$

M is certainly **equicontinuous**.

Take a sequence (f_n) in M which converges uniformly to $g \in C([a, b])$. For any $\varepsilon > 0$ there exists N s.t. $\|g - f_N\| < \varepsilon$, hence for any $x, y \in [a, b]$, by the triangle inequality

$$|g(x) - g(y)| \leq |g(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - g(y)| \leq 2\varepsilon + \sqrt{|x-y|}.$$

Since ε was arbitrary, we may conclude that $|g(x) - g(y)| \leq \sqrt{|x-y|}$ and thus $g \in M$. Therefore M is also **closed**, and thus compact.

6. We note that $L = \mathbb{R}^{n\times n} = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is a Banach ring with the norm $\|T\| = \|T\|_{\mathrm{op}}$ (see sections 4.10 and 5.2 of the lecture notes). Further, we consider $\exp\colon L \to L$,

$$\exp T := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

which converges pointwise by the majorization theorem for series (see Theorem 4.76). Thus (using commutativity)

$$f(T) = \exp\bigl(T(I-T)\bigr) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n (I-T)^n, \quad T \in L.$$

Instead of this explicit expression, let us use the properties of composition,

$$f = \exp \circ g, \qquad \text{where } g\colon L \to L,\ g(T) = T(I-T) = T - T^2,$$

and the chain rule.

We first verify that $f \in C^1(L, L)$ and then aim to use the Inverse Function Theorem (Theorem 7.9). Note that

$$g(T+H) = T+H-(T+H)^2 = T-T^2+H-TH-HT-H^2 = g(T)+g'(T)[H]+O(H^2),$$

so g is differentiable at any $T \in L$, with derivative

$$g'(T)[H] = H - TH - HT,$$

$$\left\|g'(T)[H]\right\| \leq \|H\| + \|TH\| + \|HT\| \leq (1 + 2\|T\|)\|H\|$$
$$\Rightarrow \quad \left\|g'(T)\right\|_{\mathrm{op}} \leq (1 + 2\|T\|) < \infty.$$

Further, if $T, H, K \in L$,

$$\left(g'(T+K) - g'(T)\right)[H] = H - (T+K)H - H(T+K) - H + TH + HT = -KH - HK,$$
$$\Rightarrow \quad \left\|g'(T+K) - g'(T)[H]\right\| \leq \|KH\| + \|HK\| \leq 2\|K\|\|H\|$$
$$\Rightarrow \quad \left\|g'(T+K) - g'(T)\right\|_{\mathrm{op}} \leq 2\|K\|,$$

which tends to zero if $K \to 0$. Therefore, $g \in C^1(L, L)$, and furthermore

$$g'(I)[H] = H - H - H = -H = -\mathrm{id}_L[H],$$

i.e. $g'(I) = -\mathrm{id}_L \in \mathrm{Hom}(L, L)$.

Further, we have $\exp(0 + H) = \exp H = I + H + O(H^2)$, and we have also been given the information that $\exp \in C^1(L, L)$.

Now, by the chain rule, $f \in C^1(L, L)$ and

$$f(I+H) = \exp(g(I+H)) = \exp\left(\underbrace{g(I)}_{0} + \underbrace{g'(I)[H] + O(H^2)}_{K}\right) = I + K + O(K^2)$$
$$= I + g'(I)[H] + O(H^2) + O\left(\left(g'(I)[H] + O(H^2)\right)^2\right) = I - H + O(H^2).$$

In other words, $f'(I)[H] = -H$, i.e. $f'(I) = -\mathrm{id}_L$, which is invertible with continuous inverse $f'(I)^{-1} = -\mathrm{id}_L$. Therefore, by the inverse function theorem, f has a continuously differentiable inverse locally around the point $T = I$.

In the simplified case that $n = 1$, we simply study the smooth function $f(t) = e^{t-t^2}$ with $f'(1) = -1$, therefore, by continuity of f', f is strictly decreasing—and thus invertible—on some neighborhood of $t = 1$.

7. a) The upper and lower integrals are defined using upper and lower sums,

$$\overline{\int} f = \inf_P U(f, P), \qquad \underline{\int} f = \sup_P L(f, P),$$

where the infimum and supremum are taken over all partitions P of the interval $[0, 1]$. Let $P = \{x_0 = 0, x_1, x_2, \ldots, x_N = 1\}$, $x_n < x_{n+1}$, be such a partition, then, since each interval $[x_n, x_{n+1}]$ contains irrational points arbitrarily close to x_n,

$$\begin{aligned}
U(f,P) &= \sum_{n=0}^{N-1} \left(\sup_{[x_n,x_{n+1}]} f \right) (x_{n+1}-x_n) = \sum_{n=0}^{N-1} (2-x_n)(x_{n+1}-x_n) \\
&\geq \sum_{n=0}^{N-1} \frac{1}{2} \left((2-x_n)+(2-x_{n+1})\right)(x_{n+1}-x_n) \\
&= \sum_{n=0}^{N-1} \left(2(x_{n+1}-x_n) - \frac{1}{2}(x_{n+1}^2 - x_n^2) \right) = 2(x_N - x_0) - \frac{1}{2}(x_N^2 - x_0^2) = \frac{3}{2}.
\end{aligned}$$

Similarly, using $x_n \leq (x_n + x_{n+1})/2$,

$$L(f,P) = \sum_{n=0}^{N-1} \left(\inf_{[x_n,x_{n+1}]} f \right) (x_{n+1}-x_n) = \sum_{n=0}^{N-1} x_n(x_{n+1}-x_n) \leq \frac{1}{2}.$$

Further, taking the particular partition P_N with $x_n = n/N$, we see that these bounds are realized in the limit $N \to \infty$:

$$U(f,P_N) = \sum_{n=0}^{N-1} (2-n/N)/N = 2 - \frac{N(N-1)}{2N^2} \to \frac{3}{2},$$

$$L(f,P_N) = \sum_{n=0}^{N-1} n/N^2 = \frac{N(N-1)}{2N^2} \to \frac{1}{2}.$$

Thus,

$$\overline{\int} f = \frac{3}{2} \qquad \text{and} \qquad \underline{\int} f = \frac{1}{2}.$$

Since these numbers are different, f is not Darboux/Riemann integrable.

b) Yes: $\mathbb{Q}$ is a null set (see Exercise 4.31 and Lemma 5.41), and we may thus re-define f on $\mathbb{Q}$ so that $f(x) = 2 - x$ for all $x \in [0,1]$. Then f is Darboux/Riemann integrable with $\int_0^1 f = 3/2$.

8. a) Take for example $c_n := g_n(0)$. Then, by differentiability of g_n and the mean value theorem (MVT)

$$h_n(x) = g_n(x) - g_n(0) = x g_n'(t_x), \text{ some } t_x \in (0,x),$$

so $\|h_n\| \leq \|g_n'\| \leq K$ is uniformly bounded. Further, $h_n(0) = 0$ and again by MVT, $|h_n(x) - h_n(y)| = |(x-y)h_n'(t_{x,y})| \leq K|x-y|$, so that the sequence (h_n) in $C([0,1])$ is bounded and equicontinuous. By Arzela-Ascoli (Theorem 6.25), the sequence contains a convergent subsequence $(h_{n'})$ in $C([0,1])$.

We could also have used Proposition 6.28 directly.

b) Using the result from a), consider the integral $\int_0^1 h_n = \int_0^1 g_n - c_n$. We have that $\int_0^1 g_n$ is bounded for all n, and further, taking the subsequence for which $(h_{n'})$ converges, then also $\int_0^1 h_{n'}$ converges (see Theorem 6.17), so that $c_{n'} = \int_0^1 g_{n'} - \int_0^1 h_{n'}$ is a bounded sequence of real numbers. By Bolzano-Weierstrass it contains a sub-subsequence $(c_{n''})$ which converges. Also the sub-subsequence $(h_{n''})$ converges uniformly, and thus the sum $g_{n''} = h_{n''} + c_{n''}$ converges uniformly as well.