Uppsala Universitet **Prov i matematik**

Matematiska Institutionen Reel analys, 1MA226

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Duration: 14:00 - 19:00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English but English is recommended, and should contain detailed arguments. Permitted aids: Lecture notes of this lecture or some other textbook you want (please choose only one between them), and one sheet of A4 paper (both sides) with own handwritten notes. No personal computers are allowed.

1. Give your claims to the following questions, and prove your claims.

a. Give an example of a sequence of nonempty compacts $C_1, C_2, \ldots$ of $\mathbb{R}^2$ (equipped with its standard metric) such that the union $\cup_{n=1}^{\infty} C_n$ is an open set.

b. Is it possible to arrange the set $\mathbb{Q}$ by the order of natural numbers $\mathbb{N}$ from small to large?

sol-a. Refer to exam 20200819-1.

sol-b. Refer to Problem session 1-3.

2. For any two real sequences $\{a_n\}$ and $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

and

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n+b_n),$$

provided that there is no case as the form $(\pm\infty)+(\mp\infty)$.

sol-2. Refer to Problem session 1-5.

3. Prove that the series $F(x)=\sum_{n=1}^{\infty}\frac{x^3+n}{x^2+n^3}$ converges for all $x\in\mathbb{R}$, and that the function $F:\mathbb{R}\to\mathbb{R}$ is C^1.

sol-3. Refer to exam 202006-3.

4. Suppose f is differentiable on $[a,b]$, $f(a)=0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x)=0$ for all $x \in [a,b]$.
sol-4. Refer to Rudin Page119-26.

5. Suppose that $f(x)$ is bounded on $[a,b]$. Show that if for any $\delta > 0$ there holds that f is Riemann integrable on $[a+\delta, b]$, then we have f is integrable on $[a,b]$.
sol-5. Refer to Problem session 4-1.

6. Prove the integrability of Riemann function on $[0,1]$ in Riemann sense. Riemann function is defined as follows:

$$R(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q}, \\ \frac{1}{q} & x = \frac{p}{q}, \text{ p and q are mutually coprime.} \end{cases}$$

sol-6. Refer to Problem session 4-2-2.

7. Try to use intermediate value theorem to prove that we can never find a one to one continuous map from a closed interval $[a,b]$ $(a,b \in \mathbb{R})$ to a circle, which may be defined in polar coordinates as

$$K = \{(r,\theta) \mid 0 \leq \theta < 2\pi\},$$

where $r > 0$ is fixed.
sol-7. As in the assumptions, we use

$$K = \{(r,\theta) \mid 0 \leq \theta < 2\pi\},$$

to denote a circle with fixed radius $r > 0$. Let $[a,b]$ be some closed interval in $\mathbb{R}$. So if there exists some continuous one-to-one map from $[a,b]$ to K, accordingly we can establish a one-to-one continuous map, say f, from $[a,b]$ to $[0,2\pi)$, which means at least one of a and b is mapped by f to a inner point in $[0,2\pi)$, WLOG let's make a be such point. So we have $0 < f(a) < 2\pi$. Then let's choose points θ_1 and θ_2 s.t.

$$0 < \theta_1 < f(a) < \theta_2 < 2\pi.$$

Since f is a one-to-one continuous map, we then can denote by $x_1 = f^{-1}(\theta_1)$ and $x_2 = f^{-1}(\theta_2)$, which means

$$x_1, x_2 \in (a, b],$$

and

$$0 < f(x_1) < f(a) < f(x_2) < 2\pi.$$

Then by intermediate value theorem, we have that there exists $\xi \in (x_1, x_2)$ s.t $f(\xi) = f(a)$, which contradicts with one-to-one.

8. Suppose we have the equation as

$$x^2 + y + \sin(xy) = 0,$$

where $x, y \in \mathbb{R}$.

a. Prove that in a small enough neighborhood of the point $(0, 0)$, this defines a unique and continuous function $y = y(x)$ such that $y(0) = 0$.

b. Is the function $y = y(x)$ defined in (a) differentiable at $x = 0$? Give your claims and prove it.

c. Discuss the monotonicity of function $y = y(x)$ at a neighborhood of $x = 0$.

d. Does this define a unique function $x = x(y)$ in the sufficiently small neighborhood of the point $(0, 0)$ such that $x(0) = 0$? Try to use your results in (c) to give a discussion.

sol-8.

sol-a. It's easy to verify that $F(x, y)$ satisfies the assumptions of implicit function theorem, so we can conclude the existence of continuous function $y = y(x)$ in the neighborhood of $(0, 0)$.

sol-b. Since $F'_x(x, y) = 2x + y\cos(xy)$ is continuous in the neighborhood of $(0, 0)$, we conclude the differentibility of $y(x)$, and

$$y'(x) = -\frac{2x + y\cos(xy)}{1 + x\cos(xy)}.$$

sol-c. The monotonicity is easy to conclude from the formula of $y'(x)$ when x, y are small enough.

$$x > 0, y' < 0;\ \ x < 0, y' > 0.$$

sol-d. We can not verify the non-existence of $x = x(y)$ only via implicit function theorem, since $F'(0,0) = 0$. But we can get the conclusion from analysis of (c), since in the neighborhood of $(0,0)$ function $y = y(x)$ gets the strict maximum value at $x = 0$.