Uppsala University
Department of Mathematics
Reza Mohammadpour

Final Exam, 2023-03-10
Reell analys 1MA226

Time: 08:00–13:00. The exam consists of 8 problems, each worth 5 points. Solutions should be written in English, and should contain detailed arguments. Permitted aids: The lecture notes of the course or Rudin's book (please choose only one between them), and one sheet of A4 paper (both sides) with your own handwritten notes. No calculators are allowed.

1. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.)

Solution. *Refer to Problem session 2 (7).*

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:

a) $x_n = 1 + 3^{(-1)^n} + 2^{[\frac{n}{3}]}$.
(Here $[\alpha]$ denotes the integer part of a, i.e. the largest integer $\leq \alpha$.)

b) $x_n = \sum_{k=0}^{n} 5^k (k!)^{(-1)^n}$

Solution. *Refer to Problem session 1.*

3. Let (X, d) be a metric space. The metric d is called an *ultrametric* if

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}$$

for any $x, y, z \in X$.

i) Show that the Euclidean metric on $\mathbb{R}^n$, for $n \geq 2$, is not an ultrametric.

ii) Let p be any positive prime number $p \geq 2$. For any nonzero $x \in \mathbb{Q}$, there exists a unique $n \in \mathbb{Z}$ such that $x = p^n \frac{u}{v}$ with some integers u and v indivisible by p. Set $|x|_p = n$. Show that

$$d_p(x, y) := \begin{cases} 0 & \text{when } x = y, \\ \left(\frac{1}{p}\right)^{|x-y|_p} & \text{otherwise} \end{cases}$$

is an ultrametric distance on $\mathbb{Q}$, also known as the p-adic metric.

Solution. *(i) Take* $\mathbf{x} = (1, 0, \ldots, 0), \mathbf{y} = (0, 1, 0, \ldots, 0)$*, and* $\mathbf{O} = (0, \ldots, 0)$*. Then we have* $d(\mathbf{x}, \mathbf{y}) = \sqrt{2}$*,* $d(\mathbf{x}, \mathbf{O}) = 1$*, and* $d(\mathbf{y}, \mathbf{O}) = 1$*. Hence*

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{O}), d(\mathbf{y}, \mathbf{O})\},$$

which implies that d *is not an ultrametric on* $\mathbb{R}^n$*.*

(ii)First let us show that the p-adic function d_p *is a metric on* $\mathbb{Q}$*. Note that (M1) and (M2) hold directly from the definition of* d_p*. In order to prove (M3), we will prove*

$$d_p(x, y) \leq \max\{d_p(x, z), d_p(y, z)\}$$

for any $x, y, z \in \mathbb{Q}$*. This will prove (M3) and the ultrametric property at the same time. Indeed, this will follow from the inequality*

$$\max\{d_p(x, z), d_p(y, z)\} \leq d_p(x, z) + d_p(y, z).$$

Let $x, y, z \in \mathbb{Q}$ *be three different rationals. Set* $n = |x - z|_p$*, and* $m = |z - y|_p$*. Without loss of generality, assume* $n \leq m$*. By definition of* $|\cdot|_p$*, we have* $x - z = p^n \frac{u}{v}$*, and* $z - y = p^m \frac{u^*}{v^*}$*, which implies*

$$x - y = p^n \left(\frac{u}{v} + p^{m-n} \frac{u^*}{v^*} \right) = p^n \left(\frac{uv^* + p^{m-n} vu^*}{vv^*} \right).$$

By definition of $|\cdot|_p$*, we get* $n \leq |x - y|_p$ *since* vv^* *is indivisible by* p*. Since*

$$\left(\frac{1}{p}\right)^{|x-y|_p} \leq \left(\frac{1}{p}\right)^n = \max\left\{\left(\frac{1}{p}\right)^n, \left(\frac{1}{p}\right)^m\right\}$$

$$d_p(x, y) \leq \max\{d_p(x, z), d_p(y, z)\}$$

4. Prove that the series

$$F(x) = \sum_{n=1}^{\infty} e^{-nx} \sin(n^3 x)$$

converges for all $x > 0$, and that the function $F : (0, \infty) \to \mathbb{R}$ is C^1.

Solution. *Let* $[a, b]$ *be any real interval with* $0 < a < b$*. Note that*

$$\left| e^{-nx} \sin\left(n^3 x\right) \right| \leq e^{-nx} \leq e^{-na}, \quad \forall n \in \mathbb{Z}^+, x \in [a, b].$$

Furthermore the series $\sum_{n=1}^{\infty} e^{-na}$ *converges by the ratio test. Hence by Weierstrass' M-test, we conclude that the series defining* $F(x)$ *is uniformly convergent on* $[a, b]$*.*

Hence it follows that F is well-defined and continuous in the interval $[a, b]$. Since this is true for any $0 < a < b$, it follows that F is well-defined and continuous in the whole interval $(0, \infty)$.

Next consider the series obtained by formally differentiating the series for $F(x)$ term by term, i.e.:

$$\sum_{n=1}^{\infty} e^{-nx} \left(-n \sin\left(n^3 x\right) + n^3 \cos\left(n^3 x\right)\right). \tag{1}$$

We claim that this series is uniformly convergent on any interval $[a, b]$ with $0 < a < b$. Indeed, for all $n \in \mathbb{Z}_+$and $x \in [a, b]$ we have

$$\left|e^{-nx} \left(-n \sin\left(n^3 x\right) + n^3 \cos\left(n^3 x\right)\right)\right| \le e^{-na} \left(n + n^3\right)$$

We claim that this series is uniformly convergent on any interval $[a, b]$ with $0 < a < b$. Indeed, for all $n \in \mathbb{Z}_+$and $x \in [a, b]$ we have

$$\left|e^{-nx} \left(-n \sin\left(n^3 x\right) + n^3 \cos\left(n^3 x\right)\right)\right| \le e^{-na} \left(n + n^3\right).$$

Furthermore, for any $a > 0$ we have

$$\lim_{n \to \infty} \frac{e^{-(n+1)a} \left((n+1) + (n+1)^3\right)}{e^{-na} \left(n + n^3\right)} = e^{-a} < 1,$$

and hence by the ratio test, the series $\sum_{n=1}^{\infty} e^{-na} \left(n + n^3\right)$ converges. Using the above facts in combination with Weierstrass' M-test, we conclude that the series in (1) is indeed uniformly convergent on $[a, b]$. Hence by Rudin's Thm. 7.17, we have that $F'(x)$ exists for all $x \in [a, b]$, and

$$F'(x) = \sum_{n=1}^{\infty} e^{-nx} \left(-n \sin\left(n^3 x\right) + n^3 \cos\left(n^3 x\right)\right).$$

The uniform convergence pointed out above (together with the fact that each term is a continuous function of x) implies that this function is continuous in $[a, b]$. Hence F is C^1 in $[a, b]$. Since this is true for any $0 < a < b$, we conclude that F is C^1 in the whole interval $(0, \infty)$.

5. Suppose that f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution. *Rudin's Problem 5.5. Let $\varepsilon > 0$ be given. Since $\lim_{x \to +\infty} f'(x) = 0$, there exists some $R > 0$ such that $|f'(x)| \le \varepsilon$ for all $x \ge R$. Now consider an arbitrary $x \ge R$. By the Mean Value Theorem, there exists some $t \in (x, x+1)$*

such that $f(x+1) - f(x) = f'(t)$, *i.e.* $g(x) = f'(t)$. *Now* $t > x \geq R$ *and hence* $|g(x)| = |f'(t)| \leq \varepsilon$. *We have thus proved:*

$$\forall \varepsilon > 0 : \quad \exists R > 0 : \quad \forall x \geq R : \quad |g(x)| \leq \varepsilon$$

This means that $\lim_{x \to +\infty} g(x) = 0$.

6. Compute the upper and lower integrals of the function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 2 - x, & x \in [0,1] \backslash \mathbb{Q} \\ x, & x \in \mathbb{Q} \cap [0,1] \end{cases}$$

and conclude that it is not Riemann integrable.

Solution. *The upper and lower integrals are defined using upper and lower sums,*

$$\overline{\int} f = \inf_P U(f, P), \qquad \underline{\int} f = \sup_P L(f, P),$$

where the infimum and supremum are taken over all partitions P *of the interval* $[0,1]$. *Let* $P = \{x_0 = 0, x_1, x_2, \ldots, x_N = 1\}, x_n < x_{n+1}$, *be such a partition, then, since each interval* $[x_n, x_{n+1}]$ *contains irrational points arbitrarily close to* x_n,

$$\begin{aligned} U(f, P) &= \sum_{n=0}^{N-1} \left(\sup_{[x_n, x_{n+1}]} f \right) (x_{n+1} - x_n) = \sum_{n=0}^{N-1} (2 - x_n)(x_{n+1} - x_n) \\ &\geq \sum_{n=0}^{N-1} \frac{1}{2} \left((2 - x_n) + (2 - x_{n+1}) \right) (x_{n+1} - x_n) \\ &= \sum_{n=0}^{N-1} \left(2(x_{n+1} - x_n) - \frac{1}{2} \left(x_{n+1}^2 - x_n^2 \right) \right) = 2(x_N - x_0) - \frac{1}{2} \left(x_N^2 - x_0^2 \right) = \frac{3}{2}. \end{aligned}$$

Similarly, using $x_n \leq (x_n + x_{n+1})/2$,

$$L(f, P) = \sum_{n=0}^{N-1} \left(\inf_{[x_n, x_{n+1}]} f \right) (x_{n+1} - x_n) = \sum_{n=0}^{N-1} x_n (x_{n+1} - x_n) \leq \frac{1}{2}.$$

Further, taking the particular partition P_N *with* $x_n = n/N$, *we see that these oounds are realized in the limit* $N \to \infty$ *:*

$$U(f, P_N) = \sum_{n=0}^{N-1} (2 - n/N)/N = 2 - \frac{N(N-1)}{2N^2} \to \frac{3}{2},$$

$$L(f, P_N) = \sum_{n=0}^{N-1} n/N^2 = \frac{N(N-1)}{2N^2} \to \frac{1}{2}.$$

Thus,

$$\overline{\int} f = \frac{3}{2} \quad \textit{and} \quad \underline{\int} f = \frac{1}{2}.$$

Since these numbers are different, f is not Darboux/Riemann integrable.

7. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that

$$\sup_{x\in(0,1)} |f'(x)| = M < \infty.$$

Let n be a positive integer that is greater than 1. Prove that

$$\left|\sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)dx\right| \le \frac{M}{2n}.$$

Solution. *We have*

$$\begin{aligned}\left|\sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)dx\right| &= \left|\sum_{j=0}^{n-1}\left(\frac{f(j/n)}{n} - \int_{j/n}^{(j+1)/n} f(x)dx\right)\right| \\ &\le \sum_{j=0}^{n-1}\int_{j/n}^{(j+1)/n} |f(j/n) - f(x)|dx.\end{aligned}$$

On the other hand, by the mean value theorem, we obtain that for any $x \in (j/n, (j+1)/n)$ there exists $c_x \in (j/n, x)$ such that

$$f'(c_x) = \frac{f(x) - f(j/n)}{x - j/n}.$$

By the hypothesis that f' is bounded we deduce that

$$|f(x) - f(j/n)| \le M(x - j/n), \quad \forall x \in (j/n - (j+1)/n).$$

Hence,

$$\begin{aligned}\left|\sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)dx\right| &\le \sum_{j=0}^{n-1}\int_{j/n}^{(j+1)/n} M(x - j/n)dx \\ &= M\sum_{j=0}^{n-1}\left(\frac{(j+1)^2}{2n^2} - \frac{j^2}{2n^2} - \frac{j}{n^2}\right) = M\sum_{j=0}^{n-1}\frac{1}{2n^2} = \frac{M}{2n}.\end{aligned}$$

8. Prove that there exists an open set $U \subset \mathbb{R}^2$ with $(2,e) \in U$, and C^1 functions $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$, such that $u(2,e) = 0$ and $v(2,e) = 1$, and such that for every $(x,y) \in U$, $(u(x,y), v(x,y))$ is a solution to the following system of equations:

$$\begin{cases} e^u + v = x \\ u + e^v = y. \end{cases}$$

When this holds, determine the differentials $u'(2,e)$ and $v'(2,e)$.

Solution. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map*

$$F(u,v) = (e^u + v, u + e^v).$$

Note that F is C^1. We compute:

$$[F'(u,v)] = \begin{pmatrix} e^u & 1 \\ 1 & e^v \end{pmatrix}.$$

In particular

$$[F'(0,1)] = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix},$$

which is non-singular. Hence by the Inverse Function Theorem, there exists an open set $V \subset \mathbb{R}^2$ which contains the point $(0,1)$, such that $F|_V$ is $C^1, U := F(V)$ is open, and $G := (F|_V)^{-1} : U \to V$ is C^1. By the definition of $G = (F|_V)^{-1}$ we have $F(G(x,y)) = (x,y)$ for all $(x,y) \in U$. In other words:

$$\begin{cases} e^{G_1(x,y)} + G_2(x,y) = x \\ G_1(x,y) + e^{G_2(x,y)} = y, \end{cases} \quad \forall (x,y) \in U$$

By the definition of $G = (F|_V)^{-1}$ we have $F(G(x,y)) = (x,y)$ for all $(x,y) \in U$. In other words:

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Also $G(F(0,1)) = (0,1)$, i.e. $G(2,e) = (0,1)$. This means that if we write $u = G_1 : U \to \mathbb{R}$ and $v = G_2 : U \to \mathbb{R}$ then the functions u and v have all the properties required in the problem formulation!

By the chain rule we also have $F'(G(x,y)) \cdot G'(x,y) = I$ for all $(x,y) \in U$; thus in particular $F'(0,1) \cdot G'(2,e) = I$, or in other words:

$$[G'(2,e)] = [F'(0,1)]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix}^{-1} = \frac{1}{e-1} \begin{pmatrix} e & -1 \\ -1 & 1 \end{pmatrix}.$$

But we also know

$$[G'] = \begin{pmatrix} D_1G_1 & D_2G_1 \\ D_1G_2 & D_2G_2 \end{pmatrix} = \begin{pmatrix} D_1u & D_2u \\ D_1v & D_2v \end{pmatrix}$$

Hence:

$$[u'(2,e)] = \frac{1}{e-1}(e,-1) \quad \text{and} \quad [v'(2,e)] = \frac{1}{e-1}(-1,1).$$