Regression Analysis 2022-08-17

Q2 (a) $\bar{x} = 0$. So $\sum_i (x_i - \bar{x})^2 = \sum_i x_i^2 = 2m$

$$\hat{\beta} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} = \frac{\sum_i x_i y_i}{2m} = \frac{1}{2}\left[-\bar{y}_{(1)} + \bar{y}_{(2)}\right]$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} = \frac{\bar{y}_{(1)} + \bar{y}_{(2)}}{2}.$$

(b) $$\operatorname{Cov}\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \sigma^2 (X^T X)^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{2m} & 0 \\ 0 & \frac{1}{2m}\end{bmatrix}$$

(c) $$y = \frac{\bar{y}_{(1)} + \bar{y}_{(2)}}{2} + \frac{-\bar{y}_{(1)} + \bar{y}_{(2)}}{2} x$$

Same slope and intercept

Q3 (a)
$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_{2m} \\ y_{2m+1} \\ \vdots \\ y_{4m} \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -1 & z_1 & t_1 \\ \vdots & \vdots & \vdots \\ -1 & z_{2m} & t_{2m} \\ 1 & z_{2m+1} & t_{2m+1} \\ \vdots & \vdots & \vdots \\ 1 & z_{4m} & t_{4m} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{\beta} + \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ \vdots \\ e_{4m} \end{bmatrix}}_{e}$$

(b) $\hat{\beta} = (X^T X)^{-1} X^T y$

$\operatorname{Cov}(\hat{\beta} \mid X) = \sigma^2 (X^T X)^{-1}$

(c) $$X^T X = \begin{bmatrix} 0 & & \\ \sum_{i=2m+1}^{4m} z_i - \sum_{i=1}^{2m} z_i & \sum_{i=1}^{4m} z_i^2 & \\ \sum_{i=2m+1}^{4m} t_i - \sum_{i=1}^{4m} t_i & \sum_{i=1}^{4m} t_i z_i & \sum_{i=1}^{4m} t_i^2 \end{bmatrix}$$

then we need the off diagonal entries to be 0.

Q4 (a) OLS minimizes $(Y-X\beta)^T(Y-X\beta)$

$$\hat{\beta} = (X^TX)^{-1}X^Ty$$

$$Cov(\hat{\beta}|X) = (X^TX)^{-1}X^T\Sigma X(X^TX)^{-1}$$

(b) GLS minimizes $(Y-X\beta)^T\Sigma^{-1}(Y-X\beta)$

$$\hat{\beta} = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}y$$

$$Cov(\hat{\beta}|X) = (X^T\Sigma^{-1}X)^{-1}$$

(c) GLS is BLUE

Q5 (a) $\hat{\beta} = \begin{bmatrix} 3.3 \\ 2.14 \end{bmatrix}$

(b) $\tilde{\beta} = \hat{\beta} - (X^TX)^{-1}L^T[L(X^TX)^{-1}L^T]^{-1}L\hat{\beta}$ $\quad L = [1, 1]$

$$= \begin{bmatrix} -3.72 \\ 3.72 \end{bmatrix}$$

(c) We can use t test with $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\frac{a^T\hat{\beta} / \sqrt{a^T(X^TX)^{-1}a}}{\sqrt{\hat{e}^T\hat{e}/(n-p)}} \sim t(n-p) \quad \text{where } n=6 \quad p=2$$

(d) Same t test above.

Q6 (a) $y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + e_i$ $\quad X_i = \begin{cases} 0 & i=1\cdots n \\ 1 & i=n+1,\cdots,3n \end{cases}$

$$Z_i = \begin{cases} 0 & i=1,\cdots,2n \\ 1 & i=2n+1,\cdots,3n \end{cases}$$

(b) $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2$