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TENTAMEN I MATEMATIK
Probability theory II, 1MS036
October 25, 2019, 8–13

Permitted aids: Table with probability distributions (Gut, Appendix B). Calculators are not allowed.

For grade 5 the requirement is a total of at least 32 points, for grade 4 at least 25 points and the limit to pass (grade 3) is a total of 18 points.

1. (a) Let (X, Y) be a discrete random vector. Suppose $\mathrm{E}(|Y|) < \infty$. Prove that

$$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|X)).$$

(3p)

   (b) Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed discrete random variables with $\mathrm{E}(|X_1|) < \infty$, and let $S_n = \sum_{i=1}^{n} X_i$. Show that

$$E(X_1|S_n) = \frac{1}{n}S_n.$$

(3p)

2. (a) Give the definition of moment generating function, $\psi_X(t)$, of a random variable X, and give an example of a random variable X where $\psi_X(t)$ does not exist for all t in some open interval containing zero. (2p)

   (b) Let $(X|M = m) \in \mathrm{Po}(m)$, where $M \in \Gamma(p, a)$. Find $P(X = k)$ for $k = 0, 1 \dots$, and show that X has a negative binomial distribution. (4p)

3. Let $X \in \Gamma(p_1, a)$ and $Y \in \Gamma(p_2, a)$ be independent and let $U = X + Y$ and $V = X/(X + Y)$.

   (a) Find the joint distribution of (U, V) and show that U and V are independent. (4p)

   (b) Show that $V \in \beta(p_1, p_2)$. (3p)

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4. The normal random vector (X, Y) has moment generating function

$$\Psi_{X,Y}(s,t) = e^{2s+3t+s^2+cst+2t^2},$$

where c is a constant.

(a) Determine c so that $X + 2Y$ and $2X - Y$ become independent. (4p)

(b) Let c be chosen like in (a) so that $X + 2Y$ and $2X - Y$ are independent. Express

$$P(X + 2Y < 2X - Y)$$

in terms of the distribution function of a standard normal random variable. (3p)

5. Suppose $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\mathbf{X} = (X_1, X_2, X_3)^t$, $\boldsymbol{\mu} = (1, 4, 2)^t$, and $\boldsymbol{\Sigma} = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 9 \end{pmatrix}$.
Find the conditional distribution of (X_1, X_2) given that $X_1 + X_2 + X_3 = z$. (7p)

6. Let $(X_n)_{n=1}^{\infty}$, and $(Y_n)_{n=1}^{\infty}$ be two independent sequences of independent and identically distributed random variables where $\mathrm{E}(X_1) = \mathrm{E}(Y_1) = \mu$ and $\mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = \sigma^2$, $\sigma > 0$. Let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S_X^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2,$$

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, \qquad S_Y^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2.$$

(a) Show that

$$S_n = \frac{1}{\sqrt{2}}\sqrt{S_X^2(n) + S_Y^2(n)} \xrightarrow{p} \sigma, \quad \text{as } n \to \infty.$$

(3p)

(b) Show that

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \bar{Y}_n)}{\sigma\sqrt{2}}$$

converges in distribution as $n \to \infty$, and find the limiting distribution. (2p)

(c) Show that

$$T_n = \frac{\sqrt{n}(\bar{X}_n - \bar{Y}_n)}{\sqrt{S_X^2(n) + S_Y^2(n)}}$$

converges in distribution as $n \to \infty$, and find the limiting distribution. (2p)

B

Some Distributions and Their Characteristics

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n=1,2,\ldots$, $0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^{it}}$
First success $\mathrm{Fs}(p)$, $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1-qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n=1,2,3,\ldots$, $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m>0$	$p(k)=e^{-m}\frac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N$, $N=1,2,\ldots$, $p=0,\frac{1}{N},\frac{2}{N},\ldots,1$	$p(k)=\dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x) = \frac{1}{b-a},\ a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x) = 1,\ 0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x) = \frac{1}{2},\ \lvert x\rvert < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x) = \frac{2}{b-a}\left(1 - \frac{2}{b-a}\left\lvert x - \frac{a+b}{2}\right\rvert\right)$, $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x) = 1 - \lvert x\rvert,\ \lvert x\rvert < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a > 0$	$f(x) = \frac{1}{a}\,e^{-x/a},\ x > 0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a > 0,\ p > 0$	$f(x) = \frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x > 0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n = 1,2,3,\ldots$	$f(x) = \frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x > 0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a > 0$	$f(x) = \frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty < x < \infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s > 0$	$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0 < x < 1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) - \Gamma(\beta+1)^2\big)$	$*$
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	$*$
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	$*$
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	$*$
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2},$ $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	$*$

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x)=\frac{1}{\pi}\cdot\frac{a}{a^2+(x-m)^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x)=\dfrac{1}{\pi}\cdot\dfrac{1}{1+x^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto	$f(x)=\dfrac{\alpha k^\alpha}{x^{\alpha+1}},\ x>k$	$\dfrac{\alpha k}{\alpha-1},\ \alpha>1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha>2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k>0,\ \alpha>0$				