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TENTAMEN I MATEMATIK
Probability theory II, 1MS036
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Permitted aids: Table with probability distributions (Gut, Appendix B). Calculators are not allowed.

For grade 5 the requirement is a total of at least 32 points, for grade 4 at least 25 points and the limit to pass (grade 3) is a total of 18 points.

1. (a) Let (X, Y) be a discrete random vector. Suppose $\mathrm{E}(|Y|) < \infty$. Prove that

$$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|X)).$$

(3p)

(b) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed discrete random variables with $\mathrm{E}(|X_1|) < \infty$, and let $S_n = \sum_{i=1}^n X_i$. Show that

$$E(X_1|S_n) = \frac{1}{n} S_n.$$

(3p)

Solution:

(a)

$$\begin{aligned}
\mathrm{E}(\mathrm{E}(Y|X)) &= \sum_x \mathrm{E}(Y|X=x)p_X(x) = \sum_x \sum_y y p_{Y|X=x}(y) p_X(x) \\
&= \sum_x \sum_y y \frac{p_{X,Y}(x,y)}{p_X(x)} p_X(x) = \sum_y y \underbrace{\sum_x p_{X,Y}(x,y)}_{p_Y(y)} = \mathrm{E}(Y).
\end{aligned}$$

(b) By symmetry $\mathrm{E}(X_1|S_n) = \mathrm{E}(X_2|S_n) = \ldots = \mathrm{E}(X_n|S_n)$, and since

$$S_n = \mathrm{E}(S_n|S_n) = \sum_{i=1}^n \mathrm{E}(X_i|S_n) = n\mathrm{E}(X_1|S_n),$$

it thus follows that follows that

$$\mathrm{E}(X_1|S_n) = \frac{1}{n} S_n.$$

2. (a) Give the definition of moment generating function, $\psi_X(t)$, of a random variable X, and give an example of a random variable X where $\psi_X(t)$ does not exist for all t in some open interval containing zero. (2p)

(b) Let $(X|M=m) \in \text{Po}(m)$, where $M \in \Gamma(p,a)$. Find $P(X=k)$ for $k=0,1\ldots$, and show that X has a negative binomial distribution. (4p)

Solution:

(a) The moment generating function of X is defined by

$$\psi_X(t) = \mathrm{E}(e^{tX}),$$

provided the expectation exists for all t in some open interval containing zero. If X has a Cauchy distribution, then $\psi_X(t)$ does not exist for all t in an open interval containing zero.

(b) Since $\psi_{\text{Po}(m)}(t) = e^{m(e^t-1)}$, $\psi_{\Gamma(p,a)}(t) = \frac{1}{(1-at)^p}$, and $\psi_{\text{NBin}(n,p)}(t) = (\frac{p}{(1-(1-p)e^t})^n$, see Gut Appendix B, it follows that the random variable X has moment generating function

$$\begin{aligned}\psi_X(t) &= \mathrm{E}(e^{tX}) = \mathrm{E}(\mathrm{E}(e^{tX}|M)) = \mathrm{E}(e^{M(e^t-1)}) \\ &= \psi_M(e^t-1) = \frac{1}{(1-a(e^t-1))^p} = \left(\frac{1/(1+a)}{1-\frac{a}{1+a}e^t}\right)^p.\end{aligned}$$

Assuming p is a positive integer it thus follows from the uniqueness theorem that $X \in \text{NBin}(p, \frac{1}{1+a})$, i.e.

$$P(X=k) = \binom{p+k-1}{k}(\frac{1}{1+a})^p(\frac{a}{1+a})^k, \ k=0,1,2,\ldots$$

3. Let $X \in \Gamma(p_1,a)$ and $Y \in \Gamma(p_2,a)$ be independent and let $U = X+Y$ and $V = X/(X+Y)$.

(a) Find the joint distribution of (U,V) and show that U and V are independent. (4p)

(b) Show that $V \in \beta(p_1,p_2)$. (3p)

Solution:

If $U = X+Y$ and $V = X/(X+Y)$, then $X = UV$ and $Y = U-UV = U(1-V)$. From the transformation theorem, independence, and since $f_{\Gamma(p,a)}(x) = \frac{1}{\Gamma(p)}x^{p-1}\frac{1}{a^p}e^{-x/a}, \ x>0,$

it follows that

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(uv, u(1-v)) \cdot \underbrace{\left| \det \left(\begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix} \right) \right|}_{|-u|} \\
&= f_X(uv) f_Y(u(1-v)) u \\
&= f_{\Gamma(p_1,a)}(uv) f_{\Gamma(p_2,a)}(u(1-v)) u \\
&= \frac{1}{\Gamma(p_1)} (uv)^{p_1-1} \frac{1}{a^{p_1}} e^{-uv/a} \frac{1}{\Gamma(p_2)} (u(1-v))^{p_2-1} \frac{1}{a^{p_2}} e^{-(u(1-v))/a} u \\
&= \frac{1}{\Gamma(p_1)\Gamma(p_2)} u^{p_1+p_2-1} e^{-u/a} \frac{1}{a^{p_1} a^{p_2}} v^{p_1-1} (1-v)^{p_2-1} \\
&= \underbrace{\left(\frac{1}{\Gamma(p_1+p_2)} u^{p_1+p_2-1} \frac{1}{a^{p_1+p_2}} e^{-u/a} \right)}_{f_{\Gamma(p_1+p_2,a)}(u)} \underbrace{\left(\frac{\Gamma(p_1+p_2)}{\Gamma(p_1)\Gamma(p_2)} v^{p_1-1} (1-v)^{p_2-1} \right)}_{f_{\beta(p_1,p_2)}(v)}, \\
& \quad u > 0,\ 0 < v < 1.
\end{aligned}
$$

Thus $U \in \Gamma(p_1 + p_2, a)$ and $V \in \beta(p_1, p_2)$ are independent.

4. The normal random vector (X, Y) has moment generating function

$$\Psi_{X,Y}(s,t) = e^{2s+3t+s^2+cst+2t^2},$$

where c is a constant.

(a) Determine c so that $X + 2Y$ and $2X - Y$ become independent. (4p)

(b) Let c be chosen like in (a) so that $X + 2Y$ and $2X - Y$ are independent. Express

$$P(X + 2Y < 2X - Y)$$

in terms of the distribution function of a standard normal random variable. (3p)

Solution:

$$\Psi_{X,Y}(s,t) = e^{2s+3t+s^2+cst+2t^2} = e^{\boldsymbol{t}^t \boldsymbol{\mu} + \frac{1}{2} \boldsymbol{t}^t \boldsymbol{\Sigma} \boldsymbol{t}},$$

where $\boldsymbol{t}^t = (s, t)$, $\boldsymbol{\mu} = (2, 3)^t$, and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 2 & c \\ c & 4 \end{pmatrix}.$$

Thus if $\boldsymbol{X} = (X, Y)^t$, then $\boldsymbol{X} \in \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

If $U = X + 2Y$ and $V = 2X - Y$ then

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

is a normal random vector with covariance matrix

$$\underbrace{\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}2 & c\\ c & 4\end{pmatrix}}_{\begin{pmatrix}2+2c & c+8\\ 4-c & 2c-4\end{pmatrix}}\underbrace{\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}^t}_{\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}} = \begin{pmatrix}18+4c & 3c-4\\ 3c-4 & 12-4c\end{pmatrix}.$$

It follows that U and V are independent if and only if $3c-4=0$, i.e. if and only if $c=4/3$.

Since $P(X+2Y<2X-Y)=P(3Y-X<0)$ and since

$$3Y-X\in \mathrm{N}\left(\begin{pmatrix}-1 & 3\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}, \begin{pmatrix}-1 & 3\end{pmatrix}\begin{pmatrix}2 & c\\ c & 4\end{pmatrix}\begin{pmatrix}-1\\ 3\end{pmatrix}\right)\in \mathrm{N}(7,38-6c)\underbrace{=}_{c=4/3}\mathrm{N}(7,30),$$

it follows that

$$P(3Y-X<0)=\Phi(-7/\sqrt{30}),$$

where Φ denotes the distribution function of a standard normal random variable.

5. Suppose $\mathbf{X}\in N(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\mathbf{X}=(X_1,X_2,X_3)^t$, $\boldsymbol{\mu}=(1,4,2)^t$, and $\boldsymbol{\Sigma}=\begin{pmatrix}4 & -2 & -2\\ -2 & 4 & -2\\ -2 & -2 & 9\end{pmatrix}$.
Find the conditional distribution of (X_1,X_2) given that $X_1+X_2+X_3=z$. (7p)

Solution: Let $Y_1=X_1$, $Y_2=X_2$ and $Y_3=X_1+X_2+X_3$. Then

$$\begin{pmatrix}Y_1\\ Y_2\\ Y_3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix},$$

is normally distributed with expectation

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 4\\ 2\end{pmatrix}=\begin{pmatrix}1\\ 4\\ 7\end{pmatrix},$$

and covariance matrix

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}4 & -2 & -2\\ -2 & 4 & -2\\ -2 & -2 & 9\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}^t=\begin{pmatrix}4 & -2 & 0\\ -2 & 4 & 0\\ 0 & 0 & 5\end{pmatrix}.$$

From the block diagonal form of the covariance matrix it therefore follows that $Y_3\in \mathrm{N}(7,5)$ is independent of

$$\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}\in \mathrm{N}\left(\begin{pmatrix}1\\ 4\end{pmatrix},\begin{pmatrix}4 & -2\\ -2 & 4\end{pmatrix}\right),$$

and thus

$$((X_1, X_2)|X_1 + X_2 + X_3 = z) \in \mathrm{N}\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}\right).$$

6. Let $(X_n)_{n=1}^{\infty}$, and $(Y_n)_{n=1}^{\infty}$ be two independent sequences of independent and identically distributed random variables where $\mathrm{E}(X_1) = \mathrm{E}(Y_1) = \mu$ and $\mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = \sigma^2$, $\sigma > 0$. Let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S_X^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2,$$

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, \qquad S_Y^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2.$$

(a) Show that

$$S_n = \frac{1}{\sqrt{2}}\sqrt{S_X^2(n) + S_Y^2(n)} \xrightarrow{p} \sigma, \quad \text{as } n \to \infty.$$

(3p)

(b) Show that

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \bar{Y}_n)}{\sigma\sqrt{2}}$$

converges in distribution as $n \to \infty$, and find the limiting distribution. (2p)

(c) Show that

$$T_n = \frac{\sqrt{n}(\bar{X}_n - \bar{Y}_n)}{\sqrt{S_X^2(n) + S_Y^2(n)}}$$

converges in distribution as $n \to \infty$, and find the limiting distribution. (2p)

Solution:

(a) From the law of large numbers it follows that $\bar{X}_n \xrightarrow{p} \mu$ and since $g(x) = x^2$ is a continuous function it follows that $g(\bar{X}_n) \xrightarrow{p} g(\mu)$, i.e. $\bar{X}_n^2 \xrightarrow{p} \mu^2$, as $n \to \infty$.
From the law of large numbers it also follows that

$$\frac{\sum_{i=1}^{n} X_i^2}{n} \xrightarrow{p} E(X_1^2) = \sigma^2 + \mu^2, \text{ as } n \to \infty.$$

Thus

$$\frac{1}{n}(\sum_{i=1}^{n} X_i^2 - n\bar{X}_n^2) = \frac{\sum_{i=1}^{n} X_i^2}{n} - \bar{X}_n^2 \xrightarrow{p} \sigma^2 + \mu^2 - \mu^2 = \sigma^2, \text{ as } n \to \infty,$$

and thus

$$
\begin{aligned}
S_X^2(n) &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i^2+\bar{X}_n^2-2X_i\bar{X}_n) \\
&= \frac{1}{n-1}(\sum_{i=1}^{n}X_i^2-n\bar{X}_n^2) = \frac{n}{n-1}(\frac{1}{n}\sum_{i=1}^{n}X_i^2-n\bar{X}_n^2) \xrightarrow{p} \sigma^2,
\end{aligned}
$$

as $n\to\infty$.
Similarly $S_Y^2(n) \xrightarrow{p} \sigma^2$, as $n\to\infty$, and thus $S_X^2(n)+S_Y^2(n) \xrightarrow{p} 2\sigma^2$, as $n\to\infty$. Finally, since $g(x)=\sqrt{x/2}$ is continuous it follows that $g(S_X^2(n)+S_Y^2(n)) \xrightarrow{p} g(\sigma^2)$ i.e.

$$S_n = \frac{1}{\sqrt{2}}\sqrt{S_X^2(n)+S_Y^2(n)} \xrightarrow{p} \sigma, \quad \text{as } n\to\infty.$$

(b) The sequence $(X_i-Y_i)_{i=1}^{\infty}$ is i.i.d. and $\mathrm{E}(X_1-Y_1)=\mathrm{E}(X_1)-\mathrm{E}(Y_1)=0$, and $\mathrm{Var}(X_1-Y_1)=\mathrm{Var}(X_1)+\mathrm{Var}(Y_1)=2\sigma^2$. It therefore follows from the central limit theorem that

$$Z_n = \frac{\sqrt{n}(\bar{X}_n-\bar{Y}_n)}{\sigma\sqrt{2}} = \frac{\sum_{i=1}^{n}(X_i-Y_i)}{\sigma\sqrt{2n}} \xrightarrow{d} Z, \text{ as } n\to\infty,$$

where $Z\in\mathrm{N}(0,1)$.

(c) The results in (a) and (b) and Slutsky's theorem gives

$$T_n = \frac{\sqrt{n}(\bar{X}_n-\bar{Y}_n)}{\sqrt{S_X^2(n)+S_Y^2(n)}} = \sigma Z_n/S_n \xrightarrow{d} Z, \text{ as } n\to\infty,$$

where $Z\in\mathrm{N}(0,1)$, since $Z_n \xrightarrow{d} Z$, and $\sigma/S_n \xrightarrow{p} 1$, as $n\to\infty$ since $g(S_n) \xrightarrow{p} g(\sigma)$ for $g(x)=\sigma/x$, since σ is a continuity point of g.