UPPSALA UNIVERSITY
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Exam in Mathematics
Probability theory II, 1MS036
October 19, 2020, 14–19

Permitted aids: Pen/Pencil and eraser. An extended version of Gut, Appendix B, is distributed with the exam. No other aids are allowed. In particular, all forms of communication (except with the course coordinator) are strictly forbidden, and calculators are not allowed.

For grade 5 the requirement is a total of at least 32 points, for grade 4 at least 25 points and the limit to pass (grade 3) is a total of 18 points.

1. Find the unique distribution of a random variable X with moments, $\mathrm{E}(X^k)$, $k \geq 1$, given by
$$\mathrm{E}(X^k) = \begin{cases} \frac{1}{1+k} & \text{if } k \text{ is even.} \\ 0 & \text{if } k \text{ is odd} \end{cases}.$$
Hint: It may be helpful to know that $\frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$. (6p)

2. Let X and Y be independent exponentially distributed random variables with mean a.
   (a) Find the conditional density of X given that $X + Y = t$, where t is a positive constant. (4p)
   (b) Find $\mathrm{Var}(X|X+Y)$. (2p)

3. Let X be an exponentially distributed random variable with mean 1. Let N be the integer part of X and D be the fractional part of X, i.e. $N = \lfloor X \rfloor$ and $D = X - N$.
   (a) Show that N and D are independent. (3p)
   (b) Find the distribution of N. (2p)
   (c) Find the distribution of D. (2p)

4. Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent $L(a)$-distributed random variables and let $S_N = \sum_{i=1}^{N} X_i$ where N is a random variable, independent of $(X_n)_{n=1}^{\infty}$, with probability generating function $g_N(t) = \frac{t}{4-3t}$, $|t| < 4/3$. Show that $\frac{S_N}{2} \in L(a)$. (7p)

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5. Suppose $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\mathbf{X} = (X_1, X_2, X_3)^t$, $\boldsymbol{\mu} = (1, 2, -1)^t$, and $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Let $Y_1 = X_1 + X_2 - 2$ and $Y_2 = X_1 + 3X_3 + 2$.

   (a) Find the joint density function of (Y_1, Y_2). (4p)

   (b) Find a constant c such that Y_1 and $X_1 + cX_3$ are independent or prove that no such constant exists. (3p)

6. Let $(X_i)_{i \geq 1}$ and $(Y_i)_{i \geq 1}$ be two independent sequences of independent discrete random variables with $P(X_i = -1) = P(X_i = 1) = P(Y_i = 1) = P(Y_i = 2) = 0.5$ for any $i \geq 1$. Show that

$$n^{-3/2} \frac{(\sum_{i=1}^n X_i)(\sum_{i=1}^n Y_i)^2}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i^2}$$

   converges in distribution as $n \to \infty$, and find the limiting distribution. (7p)

B

Some Distributions and Their Characteristics

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, characteristic and momentgenerating functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	Var X	$\varphi_X(t)$	$\psi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}	e^{ta}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$	$(e^{-t}+e^{t})/2$
Bernoulli Be(p), $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$	$q+pe^{t}$
Binomial Bin(n,p), $n=1,2,\ldots,\ 0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$	$(q+pe^{t})^n$
Geometric Ge(p), $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^{it}}$	$\frac{p}{1-qe^{t}}$
First success Fs(p), $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1-qe^{it}}$	$\frac{pe^{t}}{1-qe^{t}}$
Negative binomial NBin(n,p), $n=1,2,3,\ldots,$ $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$	$(\frac{p}{1-qe^{t}})^n$
Poisson Po(m), $m>0$	$p(k)=e^{-m}\frac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$	$e^{m(e^{t}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N,$ $N=1,2,\ldots,$ $p=0,\frac{1}{N},\frac{2}{N},\ldots,1$	$p(k)=\dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\frac{N-n}{N-1}$	*	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, characteristic and momentgenerating functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	Var X	$\varphi_X(t)$	$\psi_X(t)$
Uniform/Rectangular					
$U(a,b)$	$f(x) = \frac{1}{b-a},\ a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
$U(0,1)$	$f(x) = 1,\ 0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$	$\frac{e^{t}-1}{t}$
$U(-1,1)$	$f(x) = \frac{1}{2},\ \lvert x\rvert < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$	$\frac{e^{t}-e^{-t}}{2t}$
Triangular					
$\mathrm{Tri}(a,b)$	$f(x) = \frac{2}{b-a}\left(1 - \frac{2}{b-a}\left\lvert x - \frac{a+b}{2}\right\rvert\right)$, $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$	$\left(\frac{e^{tb/2}-e^{ta/2}}{\frac{1}{2}t(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x) = 1 - \lvert x\rvert,\ \lvert x\rvert < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$	$\left(\frac{e^{t/2}-e^{-t/2}}{t}\right)^2$
Exponential	$f(x) = \frac{1}{a}e^{-x/a},\ x > 0$	a	a^2	$\frac{1}{1-ait}$	$\frac{1}{1-at}$
$\mathrm{Exp}(a),\ a > 0$					
Gamma	$f(x) = \frac{1}{\Gamma(p)}x^{p-1}\frac{1}{a^p}e^{-x/a},\ x > 0$	pa	pa^2	$\frac{1}{(1-ait)^p}$	$\frac{1}{(1-at)^p}$
$\Gamma(p,a),\ a > 0,\ p > 0$					
Chi-square	$f(x) = \frac{1}{\Gamma(\frac{n}{2})}x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x > 0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$	$\frac{1}{(1-2t)^{n/2}}$
$\chi^2(n),\ n = 1,2,3,\ldots$					
Laplace	$f(x) = \frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty < x < \infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$	$\frac{1}{1-a^2t^2}$
$L(a),\ a > 0$					
Beta	$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}x^{r-1}(1-x)^{s-1}$, $0 < x < 1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*	*
$\beta(r,s),\ r,s > 0$					

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$	$\psi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	$*$	
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	$*$	
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$, $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$	$e^{\mu t+\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$	$e^{t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	$*$	
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})}\cdot\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}}$, $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	$*$	
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}$, $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2$, $n>4$	$*$	

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$	$\psi_X(t)$
Cauchy					
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2}, \ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$	$\nexists$
$C(0,1)$	$f(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{1+x^2}, \ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$	$\nexists$
Pareto	$f(x) = \dfrac{\alpha k^\alpha}{x^{\alpha+1}}, \ x > k$	$\dfrac{\alpha k}{\alpha - 1}, \ \alpha > 1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2}, \ \alpha > 2,$	*	
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$					