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Department of Mathematics
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Exam in Mathematics
Probability theory II, 1MS036
October 19, 2020, 14–19

Permitted aids: Pen/Pencil and eraser. An extended version of Gut, Appendix B, is distributed with the exam. No other aids are allowed. In particular, all forms of communication (except with the course coordinator) are strictly forbidden, and calculators are not allowed.

For grade 5 the requirement is a total of at least 32 points, for grade 4 at least 25 points and the limit to pass (grade 3) is a total of 18 points.

1. Find the unique distribution of a random variable X with moments, $\mathrm{E}(X^k)$, $k \geq 1$, given by

$$\mathrm{E}(X^k) = \begin{cases} \frac{1}{1+k} & \text{if } k \text{ is even.} \\ 0 & \text{if } k \text{ is odd} \end{cases}.$$

Hint: It may be helpful to know that $\frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$. (6p)

Solution: The random variable X has moment generating function

$$\begin{aligned} \psi_X(t) &= \mathrm{E}(e^{tX}) = \sum_{k=0}^{\infty} \frac{t^k \mathrm{E}(X^k)}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k} \mathrm{E}(X^{2k})}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k+1)!} \\ &= \frac{1}{t} \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \\ &= \frac{e^t - e^{-t}}{2t} = \psi_{\mathrm{U}(-1,1)}(t). \end{aligned}$$

Thus, by Gut, Theorem 3.1. page 63, $X \in \mathrm{U}(-1, 1)$.

2. Let X and Y be independent exponentially distributed random variables with mean a.

   (a) Find the conditional density of X given that $X + Y = t$, where t is a positive constant. (4p)

   (b) Find $\mathrm{Var}(X|X + Y)$. (2p)

Solution:

(a) If $Z = X+Y$, then $Z \in \Gamma(2,a)$, since X and Y have moment generating functions $\psi_X(t) = \psi_Y(t) = \frac{1}{1-at}$, $t < 1/a$, and thus Z has moment generating function

$$\psi_Z(t) = \psi_X(t)\psi_Y(t) = \frac{1}{(1-at)^2},\ t < 1/a,$$

i.e. the moment generating function of a $\Gamma(2,a)$-distributed random variable. Thus

$$\begin{aligned} f_{X|X+Y=t}(x) &= \frac{f_{X,Z}(x,t)}{f_Z(t)} = \frac{f_X(x)f_Y(t-x)}{f_Z(t)} \\ &= \frac{(a^{-1}e^{-x/a})(a^{-1}e^{-(t-x)/a})}{\frac{1}{\Gamma(2)}t^{2-1}a^{-2}e^{-t/a}}) = t^{-1}, \quad t > 0,\ 0 \le x \le t. \end{aligned}$$

Thus $(X|X+Y) \in \mathrm{U}(0, X+Y)$.

(b) Since a $\mathrm{U}(a,b)$-distributed random variable has variance $(b-a)^2/12$ we have

$$\mathrm{Var}(X|X+Y) = \frac{(X+Y)^2}{12}.$$

3. Let X be an exponentially distributed random variable with mean 1. Let N be the integer part of X and D be the fractional part of X, i.e. $N = \lfloor X \rfloor$ and $D = X - N$.

(a) Show that N and D are independent. (3p)

(b) Find the distribution of N. (2p)

(c) Find the distribution of D. (2p)

Solution: The random variable X has distribution function

$$F_X(t) = P(X \le t) = 1 - e^{-t},\ t > 0.$$

$$\begin{aligned} P(N \le n, D \le x) &= P(\cup_{k=0}^n \{k \le X \le k+x\}) \\ &= \sum_{k=0}^n P(k \le X \le k+x) \\ &= \sum_{k=0}^n (F_X(k+x) - F_X(k)) \\ &= \sum_{k=0}^n (e^{-k} - e^{-(k+x)}) \\ &= (1-e^{-x})\sum_{k=0}^n e^{-k} \\ &= \underbrace{\frac{1-e^{-x}}{1-e^{-1}}}_{P(D\le x)} \underbrace{\sum_{k=0}^n \underbrace{(e^{-1})^k(1-e^{-1})}_{P(N=k)}}_{P(N\le n)}, \quad 0 \le x \le 1,\ n \in \mathbb{Z}_+ \end{aligned}$$

so $N \in \mathrm{Ge}(1-e^{-1})$ and D has distribution function

$$F_D(t) = \begin{cases} 0, & t < 0 \\ \frac{1-e^{-t}}{1-e^{-1}}, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases},$$

and N and D are independent.

4. Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent $L(a)$-distributed random variables and let $S_N = \sum_{i=1}^{N} X_i$ where N is a random variable, independent of $(X_n)_{n=1}^{\infty}$, with probability generating function $g_N(t) = \frac{t}{4-3t}$, $|t| < 4/3$. Show that $\frac{S_N}{2} \in L(a)$. (7p)

Solution: Since the moment generating function of an $L(a)$-distributed random variable is given by $\psi_{L(a)}(t) = \frac{1}{1-a^2t^2}$, $|t| < 1/a$, and since

$$\begin{aligned}
\psi_{\frac{S_N}{2}}(t) &= \mathrm{E}(e^{t\frac{S_N}{2}}) = \mathrm{E}(\mathrm{E}(e^{t\frac{S_N}{2}}|N)) \\
&= \mathrm{E}(\mathrm{E}(e^{\frac{t}{2}\sum_{i=1}^{N} X_i}|N)) = \mathrm{E}(\mathrm{E}(\prod_{i=1}^{N} e^{\frac{t}{2}X_i}|N)) \\
&= \mathrm{E}((\psi_{X_1}(t/2))^N) \\
&= g_N(\psi_{X_1}(t/2)) = g_N(\frac{1}{1-a^2t^2/4}) \\
&= \frac{\frac{1}{1-a^2t^2/4}}{4-3\frac{1}{1-a^2t^2/4}} = \frac{1}{4(1-a^2t^2/4)-3} = \frac{1}{1-a^2t^2}, \quad |t| < 1/a,
\end{aligned}$$

and the moment generating function uniquely characterizes the distribution, it follows that $\frac{S_N}{2} \in L(a)$.

5. Suppose $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\mathbf{X} = (X_1, X_2, X_3)^t$, $\boldsymbol{\mu} = (1, 2, -1)^t$, and $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Let $Y_1 = X_1 + X_2 - 2$ and $Y_2 = X_1 + 3X_3 + 2$.

   (a) Find the joint density function of (Y_1, Y_2). (4p)

   (b) Find a constant c such that Y_1 and $X_1 + cX_3$ are independent or prove that no such constant exists. (3p)

Solution:

(a) It follows (from Gut, Theorem 3.1, p.121) that the random vector

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix},$$

is normal with mean vector

$$\boldsymbol{\mu} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}^t = \begin{pmatrix} 2 & 4 \\ 4 & 25 \end{pmatrix}.$$

Thus, (See Gut Theorem 5.1, p.125),

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi} \frac{1}{\sqrt{\det(\boldsymbol{\Sigma})}} e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})},$$

and since $\det(\boldsymbol{\Sigma}) = 50 - 16 = 34$ and $\boldsymbol{\Sigma}^{-1} = \frac{1}{34}\begin{pmatrix} 25 & -4 \\ -4 & 2 \end{pmatrix}$, it follows that

$$f_{(Y_1,Y_2)}(y_1, y_2) = \frac{1}{2\pi} \frac{1}{\sqrt{34}} e^{-\frac{1}{68}(25y_1^2 + 2y_2^2 - 50y_1 - 8y_1y_2 + 8y_2 + 25)}.$$

(b) The random vector $(Y_1, X_1 + cX_3)^t$ is normal with covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & c \end{pmatrix}^t = \begin{pmatrix} 2 & 1+c \\ 1+c & 1+2c+2c^2 \end{pmatrix},$$

so if $c = -1$ then the covariance matrix is diagonal, and thus, it follows (from Gut, Theorem 7.1, page 130) that Y_1 and $X_1 - X_3$ are independent.

6. Let $(X_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ be two independent sequences of independent discrete random variables with $P(X_i = -1) = P(X_i = 1) = P(Y_i = 1) = P(Y_i = 2) = 0.5$ for any $i \geq 1$. Show that

$$n^{-3/2} \frac{(\sum_{i=1}^n X_i)(\sum_{i=1}^n Y_i)^2}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i^2}$$

converges in distribution as $n \to \infty$, and find the limiting distribution. (7p)

Solution: We want to study the limit of

$$n^{-3/2} \frac{(\sum_{i=1}^n X_i)(\sum_{i=1}^n Y_i)^2}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i^2} = \frac{(\frac{\sum_{i=1}^n X_i}{\sqrt{n}})(\frac{\sum_{i=1}^n Y_i}{n})^2}{\frac{\sum_{i=1}^n X_i}{n} + \frac{\sum_{i=1}^n Y_i^2}{n}},$$

as $n \to \infty$. By the (weak) law of large numbers

$$\frac{\sum_{i=1}^n X_i}{n} \xrightarrow{p} \mathrm{E}(X_1) = 0,$$

$$\frac{\sum_{i=1}^n Y_i^2}{n} \xrightarrow{p} \mathrm{E}(Y_1^2) = \frac{1}{2}(1^2 + 2^2) = 5/2,$$

and

$$\frac{\sum_{i=1}^n Y_i}{n} \xrightarrow{p} \mathrm{E}(Y_1) = \frac{1}{2}(1 + 2) = 3/2,$$

as $n \to \infty$, and thus since $g(x) = x^2$ is continuous is follows that

$$g(\frac{\sum_{i=1}^n Y_i}{n}) \xrightarrow{p} g(\mathrm{E}(Y_1)),$$

i.e.

$$(\frac{\sum_{i=1}^n Y_i}{n})^2 \xrightarrow{p} (\mathrm{E}(Y_1))^2 = (3/2)^2 = 9/4,$$

as $n \to \infty$. Since $\mathrm{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$ it also follows from the central limit theorem that

$$\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \xrightarrow{d} \mathrm{N}(0, 1),$$

as $n \to \infty$. By using Slutsky's theorem we therefore get that

$$(\frac{\sum_{i=1}^n X_i}{\sqrt{n}})(\frac{\sum_{i=1}^n Y_i}{n})^2 \xrightarrow{d} \frac{9}{4}Z,$$

as $n \to \infty$, where $Z \in \mathrm{N}(0, 1)$, and (by Gut, Theorem 6.2, p. 167)

$$\frac{\sum_{i=1}^n X_i}{n} + \frac{\sum_{i=1}^n Y_i^2}{n} \xrightarrow{p} 5/2,$$

as $n \to \infty$, which (by using Slutsky's theorem) gives that

$$\begin{aligned} n^{-3/2}\frac{(\sum_{i=1}^n X_i)(\sum_{i=1}^n Y_i)^2}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i^2} &= \frac{(\frac{\sum_{i=1}^n X_i}{\sqrt{n}})(\frac{\sum_{i=1}^n Y_i}{n})^2}{\frac{\sum_{i=1}^n X_i}{n} + \frac{\sum_{i=1}^n Y_i^2}{n}} \\ &\xrightarrow{d} \frac{\frac{9}{4}Z}{\frac{5}{2}} = \frac{9}{10}Z \in \mathrm{N}(0, 0.81), \end{aligned}$$

as $n \to \infty$.