Uppsala University
Department of Mathematics
February 18, 2024

1MS036
Exam

Exam in Probability II
February 18, 2024

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Help allowed at the exam: Table with probability distributions (Appendix B). Calculators are not allowed.
Grades: Maximum score is 40 points. The following limits hold for the different grades:

3	4	5
18	25	32

All motivations should be clear and easy to follow.

Problem 1

a) Let X and Y be random variables. Suppose that $E(|Y|) < \infty$. State a relationship between $E(Y|X)$ and $E(Y)$. You don't have to prove this. (1 p)

b) Suppose that the number of children in a family is a random variable X with $E(X) = \mu$, and given $X = n$ for $n \geq 1$, each of the n children in the family is a girl (independently) with probability p and a boy with probability $1 - p$. (Note that the number of girls could be written as $G = 1\{A_1\} + 1\{A_2\} + \cdots + 1\{A_n\}$ where A_i is the event that child i is a girl.) Show that $E(G) = p \cdot \mu$. (3 p)

Problem 2

a) Define convergence in probability $X_n \xrightarrow{p} X$ of a sequence $X_n, \quad n \geq 1$ of random variables. (1 p)

b) Define convergence in distribution $X_n \xrightarrow{d} X$ of a sequence $X_n, \quad n \geq 1$ of random variables. (1 p)

c) Let $X_k, \quad k \geq 1$ be independent $N(0,1)$ distributed random variables. Show that

$$\frac{X_1}{(\frac{1}{n}\sum_{k=1}^{n} X_k^2)^{1/4}} \xrightarrow{d} N(0,1) \quad \text{as} \quad n \to \infty.$$

. (3 p)

Problem 3

Let X and Y be independent random variables with a multivariate distribution

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Decide the functions $E(Y \mid X = x)$ and $E(X \mid Y = y)$. (7 p)

Problem 4

Let X and Y be independent random variables with a multivariate distribution

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Decide the distribution of $2X + 3Y$. (7 p)

Problem 5

a) Give one definition of when a random vector X is multivariate normally distributed. (You can choose any of the definitions that were discussed in the lectures.) (2 p)

b) Suppose that the random vector X has a multivariate normal distribution with expected value μ (which is a vector) and covariance matrix Σ i.e., $X \sim N(\mu, \Sigma)$. Suppose that the random variable $Y = BX + b$ for some matrix B and some vector b. What is the distribution of Y? (1 p)

c) Let Z_1, Z_2, Z_3, Z_4 be independent and $N(0,1)$ distributed. Let

$$X_1 = Z_1 + Z_2, \quad X_2 = Z_2 + Z_3, \quad X_3 = Z_3 + Z_4.$$

Decide the distribution for (X_1, X_2, X_3). Which pairs of the X_i-random variables are dependent respectively independent? (6 p)

Problem 6

Let $X_1, X_2, \ldots,$ be independent random variables with finite expected values $E(X_k) = \mu_k$ and finite variances $Var(X_k) = \sigma_k^2$. Suppose that

$$\frac{1}{n}\sum_{k=1}^{n} \mu_k \to \mu \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} \sigma_k^2 \to \sigma^2 \quad \text{as} \quad n \to \infty,$$

where $\mid \mu \mid < \infty$ and $\sigma^2 < \infty$. Show that the weak law of large numbers hold i.e., for the sum $S_n = \sum_{k=1}^{n} X_k$ we have $\frac{S_n}{n} \xrightarrow{p} \mu$ as $n \to \infty$.

(8 p)

Good luck!