Uppsala University
Department of Mathematics
October 25, 2024

1MS036
Exam

Exam in Probability II
October 25, 2024

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Help allowed at the exam: Table with probability distributions (Appendix B). Calculators are not allowed.
Grades: Maximum score is 40 points. One could also get 3 additional bonus points in total from the hand-in assignments. The following limits hold for the different grades:

3	4	5
18	25	32

All motivations should be clear and easy to follow.

Problem 1

a) Formulate the weak law of large numbers. (1 p)

b) Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with expected value 0 and finite variance σ^2. Define a sequence $Y_1, Y_2, \ldots,$ by the relationship

$$Y_n = 3\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}}.$$

One can show that Y_n converges in distribution as $n \to \infty$. Determine the distribution it converges towards.

(5 p)

Problem 2

Let $X \in Exp(1/2)$ and $Y \in Exp(1)$. Suppose that X and Y are independent. Let $U = X + Y$ and $V = \frac{X}{X+Y}$.

a) Determine the simultaneous (multivariate) density function for (U, V). (3 p)

b) Determine the marginal density functions for U and V and decide if U and V are independent. (3 p)

Problem 3

a) Suppose that X is a random variable that takes value 0 with probability $\frac{1}{2}$, value 15 with probability $\frac{1}{3}$ and value 21 with probability $\frac{1}{6}$. Determine the probability generating respectively moment generating function for X. (2 p)

b) Suppose that Y is a random variable such that

$$E(Y^k) = \frac{1}{9} + 3^{k-1}, \quad k = 1, 2, \ldots$$

Determine the distribution function for Y and show that it is unique. (5 p)

Problem 4

Let X and Y be independent random variables with a multivariate density

$$f(x,y) = \begin{cases} cx(1-y)^2 & 0 < x < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a suitable constant.

a) Determine the constant c. (1 p)

b) Determine the marginal densities $f_X(x)$ and $f_Y(y)$. (2 p)

c) Determine the regression functions (i.e., conditional expected values) $E(Y \mid X = x)$ and $E(X \mid Y = y)$. (3 p)

Problem 5

The random multivariate vector $\mathbf{X} = (X_1, X_2, X_3)$ has a 3-dimensional multivariate normal distribution with expected value $\mu = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and covariance matrix

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{pmatrix},$$

respectively. Determine the conditional distribution of X_1 given that $X_1 = -X_2$. (7 p)

Problem 6

A miner is trapped in a mine with three different doors. One leads to the freedom after walking for one hour, one leads him to a path which takes him back to the mine after three hours walk, and one leads to a path which takes him back to the mine after five hours walk. Suppose that he wants to get out and consequently chooses one of the doors at random, with equal probability (and does not learn anything after previous attempts).

a) Determine the probability generating function for the time it takes him to reach freedom. (5 p)

b) Determine the expected value and variance for the time it takes him to reach freedom. (3 p)

Hint: Use that the number of failed paths has a geometric distribution. (Remember that a geometric distribution is the probability distribution for the number of Bernoulli trials that have to be performed before one gets a successful trial.) Also note that once the miner chooses the right door it still takes him one more hour until he reaches freedom.

Good luck!