Uppsala University 1MS036
Department of Mathematics Exam
February 15, 2025

Exam in Probability II
February 15, 2025

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Help allowed at the exam: Table with probability distributions (Appendix B). Calculators are not allowed.
Grades: Maximum score is 40 points. The following limits hold for the different grades:

3	4	5
18	25	32

All motivations should be clear and easy to follow.

Problem 1

Let $X_1, X_2, \dots X_n$ be independent and equally distributed random variables distributed as X. Let N be a nonnegative and integer valued random variable independent of $X_1, X_2, \dots, X_n$. Suppose that $E(N) < \infty$ and $E(|X|) < \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$.

a) Express a formula for $E(S_N)$. (1 p)

b) During a Christmas day at the amusement park Liseberg there is a Poisson number of people $Po(\lambda)$, where $\lambda = 500$ who try to throw boots in a bucket. For each boot that a person throws the chance to hit is 11 %. Suppose that each person throws 7 boots. What is the expected value and variance for the number of boots in the bucket during the day? *Hint: When $Var(N) < \infty$ and $Var(X) < \infty$, we have*

$$Var(S_N) = E(N)Var(X) + (E(X))^2 Var(N).$$

(4 p)

Problem 2

The three-dimensional continuous random variable (X, Y, Z) has simultaneous density

$$f(x, y, z) = 9z^2(x + y), \quad 0 < x < 1 - y; \quad 0 < y < 1; \quad 0 < z < 1.$$

a) Determine the conditional distribution for X given $Y = y$. (3 p)

b) Determine $E(X|Y)$ and $P(E(X|Y) \leq \frac{1}{3})$. (4 p)

Problem 3

Let $X_1, X_2, \dots X_n$ be positive independent and equally distributed random variables with mean $\mu > 0$ and finite variance $\sigma^2 > 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and show that

a)

$$\sqrt{S_n} - \sqrt{n\mu} \to \quad N(0, b^2), \quad n \to \infty$$

and determine the constant b. (5 p)

b) Is it possible if we let $X_1, X_2, \dots X_n$ be positive independent and equally distributed random variables with mean $\mu = 0$ that the statement in a) is still true? (2 p)

Problem 4

Let N and $X_1, X_2, \ldots$ be independent random variables with $N \sim Po(\mu)$ and $X_i \sim Po(\lambda)$. Let $S_n = X_1 + X_2 + \cdots + X_n$.

a) Determine the probability generating function for S_N. (3 p)

b) Determine $P(S_N = 0)$ and $P(S_N = 1)$. (2 p)

c) Determine $E(S_N)$ and $Var(S_N)$ for $\mu = 5$ and $\lambda = 8$. (2 p)

Problem 5

Let $Y_1 = 3X_1 - 2X_2 + X_3$ and $Y_2 = X_1 + X_2 + X_3 - 5$, where $\mathbf{X} = (X_1, X_2, X_3)'$ has a 3-dimensional multivariate normal distribution with mean vector $\mu = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and covariance matrix

$$\mathbf{\Lambda} = \begin{pmatrix} 4 & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix},$$

respectively.

a) Determine the mean vector and covariance matrix for $Y = (Y_1, Y_2)'$. (3 p)

b) Determine $P(Y_1 \leq 0 \mid Y_2 = -2)$. Express the answer with help of the distribution function ϕ for a standard normal distribution. (4 p)

Problem 6

Let X and Y be independent random variables so that $X \in U(0,1)$ and $Y \in U(0,\alpha)$, where we assume that $\alpha \geq 1$. Determine the density function $f_Z(z)$ for $Z = X + Y$ for all values of z. Note that there could be several different cases depending on how z relates to α. (7 p)

Good luck!