Uppsala University
Department of Mathematics
August 20, 2025

1MS036
Exam

Exam in Probability II
August 20, 2025

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Help allowed at the exam: Table with probability distributions (Appendix B). Calculators are not allowed.
Grades: Maximum score is 40 points. The following limits hold for the different grades:

3	4	5
18	25	32

All motivations should be clear and easy to follow.

Problem 1

a) Suppose that $E(|Y|) < \infty$. Write a formula that describes the relationship between $E(Y|X)$ and $E(Y)$. No proof is needed. (1 p)

b) Suppose that the number of times a toy machine in the amusement park Gröna Lund in Stockholm that is played during one day (a game is to try to pick a toy by using the machine) is a random variable X with $E(X) = \mu$, and given $X = n$ for $n \geq 1$, each turn of the totally n turns is a winning game (that means a toy is picked) with probability p and a lost game with probability $1 - p$. Let V be the number of toys picked up in a day. (Note that V can be written as $V = 1\{I_1\} + 1\{I_2\} + \cdots + 1\{I_X\}$ where I_i is the event that the ith game gives a toy.) Show that $E(V) = p \cdot \mu$.

(3 p)

Problem 2

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables as X. Let N be non-negative and integer valued random variable independent of $X_1, X_2, \ldots$. Suppose that $E(N) < \infty$ och $E(|X|) < \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$.

a) Write a formula for $E(S_N)$. No proof is needed. (1 p)

b) Suppose also that $Var(N) < \infty$ and $Var(X) < \infty$ and write a formula for $Var(S_N)$. No proof is needed. (1 p)

c) During one random day at Gröna Lund, there is a Poisson number of people $Po(\lambda)$ where $\lambda = 300$ who try to throw boots in a bucket. One has counted that for every boot a person throws, the chance to hit the bucket is in mean 8 percent. Suppose every player throws 5 boots. What is the expected value and variance for the number of boots that is placed in the bucket during a day? (4 p)

Problem 3

The random vector $\mathbf{X} = (X_1, X_2, X_3)$ has a 3-dimensional normal distribution with expected value $\mu = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and covariance matrix

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

respectively. Decide the conditional distribution of $X_1 + X_3$ given $X_2 = 2$. (8 p)

Problem 4

Let X and Y be independent random variables with a multivariate distribution

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Decide the functions $E(Y \mid X = x)$ and $E(X \mid Y = y)$. (7 p)

Problem 5

Let X and Y be independent random variables with a multivariate distribution

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Decide the distribution of $2X + 3Y$. (7 p)

Problem 6

a) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with finite expectation $\mu \neq 0$ and finite variance σ^2, let $S_n = X_1 + X_2 + \cdots + X_n$ and show that

$$\frac{S_n - n\mu}{S_n + n\mu}$$

converges in probability when $n \to \infty$ and decide the limit. (3 p)

b) Show that

$$\sqrt{n}\frac{S_n - n\mu}{S_n + n\mu} \xrightarrow{d} N(0, a^2)$$

when $n \to \infty$ and decide the constant a^2. (5 p)

Good luck!