Uppsala University
Department of Mathematics
October 24, 2025

1MS036
Exam

# Exam in Probability II
# October 24, 2025

*Examinator:* Cecilia Holmgren, cecilia@math.uu.se.
*Help allowed at the exam:* Table with probability distributions (Appendix B). Calculators are not allowed.
*Grades:* Maximum score is 40 points. One could also get 3 additional bonus points in total from the hand-in assignments. The following limits hold for the different grades:

| 3 | 4 | 5 |
|---|---|---|
| 18 | 25 | 32 |

All motivations should be clear and easy to follow.

## Problem 1

a) Define the concepts convergence in distribution ($\xrightarrow{d}$) and convergence in probability ($\xrightarrow{p}$) for a sequence of random variables. (3 p)

b) Toss a symmetric coin and let $X = 1$ if one gets head and $X = 0$ if one gets tail. Let $X_1, X_2, \ldots,$ be a sequence of random variables, where $X_{2n} = X$ and $X_{2n-1} = 1 - X$ for $n = 1, 2, \ldots,$ Show that $X_k \xrightarrow{d} X$ but $X_k \not\xrightarrow{p} X$ as $k \to \infty$, i.e., convergence in distribution holds but convergence in probability does not hold. (4 p)

## Problem 2

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $X_i \sim Po(\lambda)$. Let $N$ be an independent random variables of all $X_i$, where $N \sim Po(\mu)$. Consider the sum $S_N = X_1 + \cdots + X_N$.

a) Determine the probability generating function for $S_N$. (2 p)

b) Determine $P(S_N = 0)$ and $P(S_N = 1)$. (2 p)

c) Determine $E(S_N)$ and $Var(S_N)$. (2 p)

*Hint: You may want to apply the general formula for the variance:* $Var(Y) = E(Var(Y \mid X) + Var(E(Y \mid X)$.

## Problem 3

Let $X$ and $Y$ be independent random variables, where $X \in Exp(\frac{1}{a})$ and $Y$ has density function $f_Y(y) = e^{-(y-b)}$, $y \geq b$. Let

$$U = aX + Y, \quad V = aX - Y.$$

Determine the joint density $f_{U,V}(u, v)$ as well as the marginal densities $f_U(u)$ and $f_V(v)$. Are $U$ and $V$ independent?

(6 p)

## Problem 4

Let $\mathbf{Z} = (Z_1, Z_2, Z_3)^T$, where $Z_i$ are independent $N(0,1)$ random variables. Let $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ -2 & 0 & 3 \end{pmatrix}.$$

Let $\mathbf{X} = (X_1, X_2, X_3)^T = BZ + b$ (the notation $^T$ in $\mathbf{Z}$ and $\mathbf{X}$ denotes they are column vectors). Determine the conditional distribution of $X_2$ given $X_1 = x$.

(7 p)

## Problem 5

Let $X \sim Bin(n,p)$. If $p$ is unknown it is usually estimated as $\hat{p}_n = \frac{X}{n}$. Show that

$$Y_n = \frac{\hat{p}_n - p}{\sqrt{\hat{p}_n(1-\hat{p}_n)/n}}$$

converges in distribution as $n \to \infty$ and determine the limit distribution. *Hint: Use that a binomial distribution can be regarded as a sum of i.i.d. Bernoulli random variables.* (7 p)

## Problem 6

Suppose that the $n$-dimensional random vector $X$ has moment generating function

$$\psi_X(t) = \exp\{\sum_{i=1}^{n} t_i^2 + \sum_{1 \le i < j \le n} t_i t_j\}$$

.

a) Determine the distribution for $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. *Hint: Use that the moment generating function of a multivariate random variable is defined as* $\psi_X(t_1, t_2, \ldots, t_n) = \psi_X(t) = E(exp\{t^T X\})$.
(4 p)

b) Examine if $Y_n$ converges in distribution and determine in that case the limit distribution.
(3 p)

Good luck!