Uppsala University 1MS036
Department of Mathematics Exam
February 21, 2026

# Exam in Probability II
# February 21, 2026

*Examinator:* Cecilia Holmgren, cecilia@math.uu.se.
*Help allowed at the exam:* Table with probability distributions (Appendix B). Calculators are not allowed.
*Grades:* Maximum score is 40 points. The following limits hold for the different grades:

| 3 | 4 | 5 |
|---|---|---|
| 18 | 25 | 32 |

All motivations should be clear and easy to follow.

## Problem 1

Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with expected value $\mu$ and variance $\sigma^2$.

a) Define convergence in distribution ($\xrightarrow{d}$). (2 p)

b) Let $S_n = \sum_{k=1}^{n} X_k$ and suppose the random variables $X_i$ are positive, show that

$$\frac{S_n - n\mu}{\sqrt{S_n}} \xrightarrow{d} N(0, \frac{\sigma^2}{\mu}) \text{ as } n \to \infty.$$

(4 p)

## Problem 2

a) Define convergence in probability ($\xrightarrow{p}$). (2 p)

b) Toss a symmetric coin and let $X = 1$ if one gets head and $X = 0$ if one gets tail. Let $X_1, X_2, \ldots,$ be a sequence of random variables where $X_{2n} = X$ and $X_{2n-1} = 1 - X$ for $n = 1, 2, \ldots,$. Show that $X_k \xrightarrow{d} X$ but $X_k \xrightarrow{p} X$ does not hold as $k \to \infty$. (4 p)

## Problem 3

Let the vector $(X_1, X_2)'$ be a multivariate normal distributed random variable with $E(X_1) = E(X_2) = Var(X_1) = 1$, $Var(X_2) = 2$ and $Cov(X_1, X_2) = 1$. Define $Y_1 = X_1 - 2X_2 + 1$ and $Y_2 = 2X_1 - X_2 - 1$.

a) Determine the distribution of the vector $(Y_1, Y_2)'$ and calculate $P(Y_1 \geq 0)$. (4 p)

b) Determine the conditional distribution for $Y_1$ given that $Y_2 = 0$. (3 p)

## Problem 4

Suppose that $X$ and $Y$ are independent random variables $X \in \beta(a, b)$ and $Y \in \Gamma(c, 1)$. Let $U = XY$, $V = (1 - X)Y$.

a) Calculate the simultaneous density for $(U, V)$. (5 p)

b) How should $a$, $b$ and $c$ be related for $U$ and $V$ to be independent and which distributions should $U$ and $V$ have in that case? (2 p)

## Problem 5

We assume $(X \mid M = m) \in Po(m)$ and $M \in \Gamma(p, a)$ where $Po(m)$ is a Poisson distribution with parameter $m$ and $\Gamma(p, a)$ is a Gamma distribution with parameters $(p, a)$.

a) Determine $P(X = k)$ for $k = 0, 1, 2, \ldots$, and show that $X$ has a negative binomial distribution and determine the parameters in this distribution.

(3 p)

b) Show that $(M \mid X = k) \in \Gamma(p + k, \frac{a}{1+a})$.

(4 p)

## Problem 6

A game is played as follows: A coin is tossed until head comes up. Then a dice is tossed. If the dice shows 6, the game is ended. Otherwise the coin is tossed until head comes up. Then the dice is tossed again. This is repeated until the dice shows 6 for the first time. Determine the distribution for the total number of tossed coins. (7 p)

*Hint: A probability generating function can be used.*

Good luck!