

Duration: 14:00 - 19:00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English but English is recommended, and should contain detailed arguments. Permitted aids: Lecture notes of this lecture or some other textbook you want (please choose only one between them), and one sheet of A4 paper (both sides) with own handwritten notes. No personal computers are allowed.

1. Give your claims to the following questions, and prove your claims.
 - a. Give an example of a sequence of nonempty compacts C_1, C_2, \dots of \mathbb{R}^2 (equipped with its standard metric) such that the union $\cup_{n=1}^{\infty} C_n$ is an open set.
 - b. Is it possible to arrange the set \mathbb{Q} by the order of natural numbers \mathbb{N} from small to large?

2. For any two real sequences $\{a_n\}$ and $\{b_n\}$, prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n,$$

and

$$\liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n \leq \liminf_{n \rightarrow \infty} (a_n + b_n),$$

provided that there is no case as the form $(\pm\infty) + (\mp\infty)$.

3. Prove that the series $F(x) = \sum_{n=1}^{\infty} \frac{x^3+n}{x^2+n^3}$ converges for all $x \in \mathbb{R}$, and that the function $F : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 .

4. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

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5. Suppose that $f(x)$ is bounded on $[a, b]$. Show that if for any $\delta > 0$ there holds that f is Riemann integrable on $[a + \delta, b]$, then we have f is integrable on $[a, b]$.

6. Prove the integrability of Riemann function on $[0, 1]$ in Riemann sense. Riemann function is defined as follows:

$$R(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q}, \\ \frac{1}{q} & x = \frac{p}{q}, p \text{ and } q \text{ are mutually coprime.} \end{cases}$$

7. Try to use intermediate value theorem to prove that we can never find a one to one continuous map from a closed interval $[a, b]$ ($a, b \in \mathbb{R}$) to a circle, which may be defined in polar coordinates as

$$K = \{(r, \theta) \mid 0 \leq \theta < 2\pi\},$$

where $r > 0$ is fixed.

8. Suppose we have the equation as

$$x^2 + y + \sin(xy) = 0,$$

where $x, y \in \mathbb{R}$.

- Prove that in a small enough neighborhood of the point $(0, 0)$, this defines a unique and continuous function $y = y(x)$ such that $y(0) = 0$.
- Is the function $y = y(x)$ defined in (a) differentiable at $x = 0$? Give your claims and prove it.
- Discuss the monotonicity of function $y = y(x)$ at a neighborhood of $x = 0$.
- Does this define a unique function $x = x(y)$ in the sufficiently small neighborhood of the point $(0, 0)$ such that $x(0) = 0$? Try to use your results in (c) to give a discussion.

Good luck !!!